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Scaling, Renormalization and Statistical Conservation Laws in the Kraichnan Model of Turbulent Advection

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We present a systematic way to compute the scaling exponents of the structure functions of the Kraichnan model of turbulent advection in a series of powers of ξ, adimensional coupling constant measuring the degree of roughness of the advecting velocity field. We also investigate the relation between standard and renormalization group improved perturbation theory. The aim is to shed light on the relation between renormalization group methods and the statistical conservation laws of the Kraichnan model, also known as zero modes.

KEY WORDS: Passive scalar, Anomalous scaling, renormalization group Kraichnan model, turbulence

1. INTRODUCTION

Fully developed turbulence seems to have properties that are familiar from an another branch of physics, the theory of critical phenomena. Thus certain observables are scale invariant, i.e. they exhibit power-law dependence on length scale. This power-law seems to be reasonably well captured by dimensional analysis, with however systematic discrepancies occurring that don't seem to have simple structure. This resembles the phenomena of second order phase transitions where dimensional arguments (mean field theory) do a reasonable job, but don't fully account for the true scaling exponents. Furthermore, in both cases the observed scaling exponents seem to exhibit universality, i.e. a relative independence on many details of the system: in the case of critical phenomena details of the microscopic Hamiltonian are unimportant, only symmetries matter, in the case of turbulence,

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microscopic details of the forcing mechanism that maintains the turbulent state seem irrelevant.

Important differences occur too. On the practical level, in the theory of critical phenomena a rather explicit starting point for calculating correlation functions exists in the form of Gibbs measure given in terms of an explicit Hamiltonian. This is directly accessible numerically and analytically one can gain qualitative understanding by perturbative study around the upper (and sometimes lower) critical dimension. In case of turbulence, while the dynamical equations governing fluid motion have been known for long, the analog of Gibbs distribution is not. The nature of the stationary state describing temporal averages of measurements is a dynamical problem that is unsolved. Also, there doesn't seem to be a parameter in the problem whose special value would make the dimensional analysis exact (like the dimension of space in critical phenomena) and which could then provide a basis for a perturbative study of the problem of anomalous scaling.

On the more fundamental level there also are differences. The modern renormalization group (RG) theory of critical phenomena is based on locality in position space. The effective theory of any given scale is given by a Gibbs state defined by a local Hamiltonian. The RG relates these different effective theories to each other. In turbulence the stationary state is characterized by fluxes ("cascades") of conserved quantities (energy and in 2d enstrophy). This cascade process is believed to be local in wave number space and it is not clear what the right RG description is. Both *direct*[21,25,26,50] and *inverse* (see Ref. 23 for review and also Ref. 4 for criticism) RG's have been proposed in the past. The former is analogous to the theory of critical phenomena based on locality and coarse graining in physical space whereas the latter coarse grains in wave vector space preserving locality there.

Theoretical progress in critical phenomena came in two ways: by exactly solvable models with nontrivial scaling (the 2d Ising model) and by perturbative RG analysis near a Gaussian theory (ϵ-expansion). For Navier-Stokes turbulence these options are not available. An analog of the ϵ-expansion is provided by considering random stirring where the power spectrum of the force concentrates at large wave numbers k, being proportional to $k^{4-d+2\epsilon}$. Unfortunately for small ϵ this is very different from the turbulent situation where the force is concentrated in low wave numbers. Nevertheless interesting lessons can be learned also from such small ϵ-expansions as witnessed by the renormalization group studies carried out by the St. Petersburg school (see for example Refs. 4, 51 and references therein).

A different approach which can be pursued in the investigation of fully developed turbulence is to consider phenomenological models able to capture some of the properties of turbulent fluid motion. Among such models, the Kraichnan model of passive advection[35,36] has permitted in the last years to shed some light on the mechanisms underlying the genesis of intermittency.[10,11,17,18,28,30]

An exhaustive review of the results derived from the Kraichnan model can be found in Ref. 24.

The simplifying assumption which defines the Kraichnan model is the replacement of the Navier-Stokes velocity field advecting the scalar observable with a random field, Gaussian and δ-correlated in time. This latter assumption is crucial as it ensures that the correlation functions of the scalar field satisfy closed Hopf equations that allow to relate the n-point functions to $n-2$-point functions via Green functions of differential operators built out of the spatial part of the velocity covariance. The latter involves a parameter ξ describing the smoothness of the velocity field: its realizations are Holder continuous with exponent less than $\xi/2$.

The properties of the theory versus ξ are particularly interesting. At ξ equal zero, the effect of the advection is to "renormalize" the microscopic molecular diffusivity to a macroscopic eddy-diffusivity. The resulting theory is Gaussian and provides a natural starting point for a perturbative investigation of the system in powers of ξ. Although it can be argued that the value of ξ ideally corresponding to a turbulent flow is equal to 4/3, the scalar field tends already for small values of ξ to a steady state where an inertial range sets in. Thus, the turbulent regime it is accessible in perturbation theory. This is at variance with what happens for the ϵ-expansion of the Navier-Stokes equation where the perturbative expansion has its starting point in a model with vanishing inertial range.

The scalar correlation functions were seen to have the zero molecular diffusivity limit order by order of the ξ-expansion. This result was subsequently proved rigorously for all ξ in Ref. 37. However, the main result of the ξ-perturbation theory has been the derivation of corrections to the naive scaling prediction of scaling exponents of the structure functions of the scalar(10,11,28,30) a result that also was obtained in a perturbative expansion of the structure functions in inverse powers of the spatial dimension.(17,18,44)

Both perturbative approaches were based on the study of the Hopf equations satisfied by the equal time correlations of the scalar. In the inertial range where forcing and dissipative effects are negligible, these equations reduce in the stationary state to the annihilation of the correlation function of order n by a linear operator $\mathcal{M}_n^*$. For each n, the operators $\mathcal{M}_n^*$ admit scaling zero modes that can be computed perturbatively in ξ (or in $1/d$) and can be shown to determine the leading scaling behavior of the structure functions.

The zero modes are statistically conserved quantities of the scalar field. The presence of such conservation laws provides a mechanism underlying the phenomenon of intermittency. Indeed, they were observed numerically both in passive scalar advection by a two-dimensional Navier-Stokes velocity field(16) and in shell models(6,9,58) (see also Refs. 13, 38).

The concept of zero mode has therefore proved fruitful both to shed light on properties of more realistic models of turbulence and in the analysis of the

Kraichnan model. However, an issue which was left open is how to compare the small ξ expansion of Refs. 10, 30 with the field-theoretic methods based on perturbative expansions of the Martin-Siggia-Rose functional that describes the time space time correlation functions of the theory. The interest of such question is threefold.

First, the Martin-Siggia-Rose formalism allows to study all the correlation functions described by the theory without restriction to equal times. Hence it permits to inquire what the standard perturbation theory in ξ can say about more general observables.

The second reason of interest is that the Martin-Siggia-Rose provides a natural framework to define an algorithm to compute, at least in principle, all higher orders of the perturbative expansion. In the zero mode approach, scaling exponents have been computed using scaling Ansatze, which have provided the first order in ξ term (or in some special cases exact results, see for example Ref. 52). On the contrary, using the Martin-Siggia-Rose formalism, ultraviolet renormalization and short distance expansion, expressions up to $O(\xi^3)$ of the scaling exponents were derived in Refs. 2, 3, 5.

The fact that ultra-violet renormalization proves useful in the context of the Kraichnan model might appear at first glance surprising. In general, renormalization is used to make sense of perturbative expansions affected by ultra-violet divergences. On the other hand the Kraichnan model, at least as it should be defined whenever the small ξ expansion is used, exhibits a well defined small scale behavior. It is therefore interesting to establish a precise connection between the zero-mode and the ultra-violet renormalization methods. Zero modes provide the leading inertial range asymptotics of equal time correlations. It was pointed out already in Ref. 11 that taking the Mellin transform yields a well defined way to identify the zero mode contribution given the full, all scales, expression of correlation functions. In the present paper this idea is developed in order to show how it can be implemented in principle to all orders in perturbation theory. As an example, explicit expressions of isotropic and anisotropic scaling exponents of the structure functions are obtained to second order in ξ. The result does not just recover the result found with the renormalization group in Ref. 5 but establishes the relation between the two methods and the reason why the predictions must coincide to all orders in perturbation theory.

The third reason of interest is the relationship between the different renormalization groups, direct and inverse, that have been proposed for turbulence. In particular the concept of inverse or infra-red renormalization has been proposed as an alternative tool to solve scaling in the Kraichnan model.[31,32] The special features of the Kraichnan model are probably not suited to settle the issue which of the approaches is more natural. However it provides a simple case study of what infra-red renormalization is about.

The scope of the present paper is to address in a comprehensive way the these three issues.

The paper is organized as follows. In the first Sec. 2 the passive scalar equation is introduced and the definition of the Kraichnan ensemble is given. In Sec. 3 the asymptotic expression of the velocity field in the physically relevant ranges are derived. The analysis of the correlation of the velocity field shows that the roughness parameter ξ lend itself as a parameter for a perturbative construction of the solution of the Kraichnan model.

In Sec. 4 the Hopf equations governing the dynamics of equal time correlation functions are recalled. Sec. 5 delves into the relation between the Mellin transform of the solution and statistical conservation laws. Statistical conservation laws are specified by homogeneous solutions of the Hopf equations admissible in the inertial range but not matching the boundary conditions at large spatial scales. Such solutions, referred as zero modes can be interpreted as the residues of the first poles of the Mellin transform of the full solution of the Hopf equation.

In Sec. 6 we recall the analysis of the zero modes in Ref. 11 which allows to predict the inertial range asymptotics of the structure functions of the scalar field. The prediction is encoded in a scaling Ansatz which can be tested in a perturbative expansion in powers of the Holder exponent of the velocity field.

The perturbative expansion is couched in the field theoretic formalism through the introduction of a Martin-Siggia-Rose generating function (Sec. 7). The limit of zero Holder exponent is shown to provide a Gaussian limit around which it is possible to develop a perturbative expansion according to standard Feynman rules.

In Sec. 8 the explicit expression of the scaling exponent is given up to second order in ξ. The calculation is based on the use of Mellin transform technique described in Sec. 6. Technical details of the evaluation of the integrals are deferred to appendices C, D and Ref. 53.

A consequence of the scaling Ansatz of Sec. 6 is that the scaling exponent of the structure functions also govern the blow up rate of scalar gradients versus the dissipative scale. This is recalled in Sec. 9 where the exponents are again evaluated to the second order in ξ, the evaluation being considerably easier than via the structure functions.

In Sec. 10 the Wilson's formulation of the renormalization group is recalled first in the traditional direct form and then in the inverse setup. In Sec. 11 the infrared scaling fields exhibiting the anomalous scaling of the structure functions are derived and in Sec. 13 the ultraviolet scaling fields responsible to the scaling of the scalar gradients.

The last section is devoted to conclusions. In an appendix some details of computations are collected.

2. PASSIVE ADVECTION BY A TURBULENT FLUID AND THE KRAICHNAN MODEL

The passive scalar equation describes a scalar quantity $\theta(\mathbf{x}, t)$ which is advected by a fluid moving with velocity $\mathbf{v}(\mathbf{x}, t)$ and diffuses with molecular diffusivity κ:

$$\partial_t \theta + \mathbf{v} \cdot \partial_{\mathbf{x}} \theta - \frac{\kappa}{2} \partial^2 \theta = f \tag{1}$$

The molecular diffusivity κ describes microscopic dissipative effects. The role of the force term f is to provide a source for the scalar in order to sustain the system which otherwise would decay under the effect of dissipation.

A turbulent steady state with large inertial range sets in if the forcing acts over spatial scales much larger than those where dissipation becomes relevant. The details of the forcing are supposed to be irrelevant for inertial range scaling. It is therefore convenient to chose the forcing as a Gaussian field of zero average and covariance

$$\prec f(\mathbf{x}_1, t_1) f(\mathbf{x}_2, t_2) \succ = \delta(t_2 - t_1) F\left(\frac{\mathbf{x}_1 - \mathbf{x}_2}{L_F}\right) \tag{2}$$

The spatial part of the covariance F is a smooth function decaying rapidly at infinity and satisfying

$$F(0) = F_\star > 0 \tag{3}$$

Hence the constant L_F specifies the characteristic scale (the integral scale or the correlation length) of the forcing. The delta correlation in time in (2) means that (1) is a stochastic (partial) differential equation.

In realistic models of turbulence the velocity field is specified by a solution of the incompressible Navier-Stokes equation. One would like to understand the typical behavior of the scalar given a typical realization of $\mathbf{v}$ from an ensemble of such solutions. In the Kraichnan model[36] this ensemble is replaced by an ensemble of Gaussian random velocity fields which is chosen so as to mimic some properties that are thought to be crucial of real turbulent velocity ensembles.

One considers random velocity fields with Gaussian statistics having zero average and covariance

$$\prec \mathbf{v}^\alpha(\mathbf{x}_1, t_1) \mathbf{v}^\beta(\mathbf{x}_2, t_2) \succ = \delta(t_2 - t_1) D^{\alpha\beta}(\mathbf{x}_1 - \mathbf{x}_2; m, M). \tag{4}$$

Real turbulent velocities have two characteristic length scales, the short, dissipative scale M^{-1} and the long integral scale m^{-1}. These are modeled in the Kraichnan ensemble by an ultraviolet cutoff M and infrared cutoff m in wave numbers entering the spatial part of the covariance in (4). Moreover, realistic turbulent velocities are approximately self similar for scales between the dissipative and the integral scales.

The simplest choice for the spatial part of the covariance having such properties is

$$D^{\alpha\beta}(\mathbf{x};m,M) = D_0\xi \int \frac{d^dq}{(2\pi)^d}\frac{e^{\imath\mathbf{q}\cdot\mathbf{x}}}{q^{d+\xi}}\Pi^{\alpha\beta}(\hat{\mathbf{q}})\chi_{[m,M]}(q) \tag{5}$$

where

$$\Pi^{\alpha\beta}(\hat{\mathbf{q}}) := \delta^{\alpha\beta} - \hat{\mathbf{q}}^{\alpha}\hat{\mathbf{q}}^{\beta} \tag{6}$$

with $\hat{\mathbf{q}}$ denoting the unit vector $\mathbf{q}/q$, $q = |\mathbf{q}|$ and $\chi_{[m,M]}(q)$ is the characteristic function of the interval $[m, M]$.

We consider in this paper only incompressible velocities $\partial \cdot \mathbf{v} = 0$ which is guaranteed by the tensor $\Pi^{\alpha\beta}$. It is convenient to introduce the covariance of velocity differences:

$$\prec [\mathbf{v}^{\alpha}(0,t') - \mathbf{v}^{\alpha}(x,t')][\mathbf{v}^{\beta}(0,t) - \mathbf{v}^{\beta}(\mathbf{x},t)] \succ = 2\delta(t-t')d^{\alpha\beta}(\mathbf{x};m,M) \tag{7}$$

where we have introduced the spatial velocity difference covariance

$$\begin{aligned} d^{\alpha\beta}(\mathbf{x};m,M) &= D^{\alpha\beta}(0;m,M) - D^{\alpha\beta}(\mathbf{x};m,M) \\ &= D_0\xi \int \frac{d^dq}{(2\pi)^d}\frac{1-e^{\imath\mathbf{q}\cdot\mathbf{x}}}{q^{d+\xi}}\Pi^{\alpha\beta}(\hat{\mathbf{q}})\chi_{[m,M]}(q). \end{aligned} \tag{8}$$

As seen in detail later, $d^{\alpha\beta}$ is approximatively scale invariant in the inertial range:

$$d^{\alpha\beta}(\lambda\mathbf{x};m,M) \sim \lambda^{\xi} d^{\alpha\beta}(\mathbf{x};m,M) \tag{9}$$

for

$$M^{-1} \ll |\mathbf{x}|, \lambda|\mathbf{x}|, \quad \lambda|\mathbf{x}|, |\mathbf{x}| \ll m^{-1} \tag{10}$$

The constant

$$D^{\alpha\beta}(0;m,M) = D(m^{-\xi} - M^{-\xi})\delta^{\alpha\beta} \tag{11}$$

where

$$D := D_0\frac{d-1}{d}\frac{\Omega_d}{(2\pi)^d}, \quad \Omega_d := \frac{2\pi^{d/2}}{\Gamma(d/2)} \tag{12}$$

describes the mean square velocity field which blows up as the integral scale m^{-1} tends to infinity.

Finally, the delta correlation in time of the velocity fields guarantees the statistical invariance of the velocity differences under Galilean transformations $\mathbf{v}'(\mathbf{x},t) = \mathbf{v}(\mathbf{x}+\mathbf{u}t,t) - \mathbf{u}$, an important property of the Navier-Stokes equation. More important, it leads to a relatively explicit solution of the scalar statistics.

Equation (1) together with (4) defines an infinite dimensional stochastic differential equation with multiplicative noise. In order for this object to be well defined according to the general rules of stochastic calculus,[22,47] it is necessary to specify how the product between the velocity and the scalar field in (1) is defined as the continuum limit of a finite difference stochastic equation. The Kraichnan model is supposed to be the limit of a physical system with finite time correlations. Hence it is natural to regard (1), (4) as infinite dimensional stochastic differential equation in the Stratonovich sense.

3. ASYMPTOTICS EXPRESSIONS OF THE VELOCITY ENSEMBLE

Far from being the only possible, the choice (5) suits the derivation of explicit asymptotic expressions of the velocity field. Of physical relevance is the behavior of the velocity field in the dissipative and inertial range.

3.1. Dissipative Range Asymptotics

The dissipative range is defined by the inequalities

$$mx \ll Mx \ll 1 \tag{13}$$

Under this assumption the Fourier exponential in (5) can be expanded in Taylor series. Up to leading order, the expansion yields

$$D^{\alpha\beta}(\mathbf{x}; m, M) \sim D^{\alpha\beta}(0; m, M) - \frac{D\xi M_v^{-\xi}(Mx)^2 \mathcal{T}^{\alpha\beta}(\hat{\mathbf{x}}, 2)}{(2-\xi)(d-1)(d+2)} \tag{14}$$

having introduced the rank-two real-space tensor

$$\mathcal{T}^{\alpha\beta}(\hat{\mathbf{x}}, z) = \delta^{\alpha\beta} - \frac{z}{d-1+z}\hat{\mathbf{x}}^\alpha \hat{\mathbf{x}}^\beta. \tag{15}$$

Whenever

$$\frac{m}{M} \ll 1 \tag{16}$$

the *eddy diffusivity*

$$\varkappa_\star := Dm^{-\xi} \tag{17}$$

dominates the velocity field in the dissipative range. The leading correction to the constant mode of the velocity field is smooth and vanishing as $M^{-\xi}$ tends to zero.

3.2. Inertial Range Asymptotics

The integral (8) is convergent both if the ultra-violet cut-off M is set to infinity and the infra-red cutoff m to zero. Thus, the statistics of the velocity differences exists in such limit. It can be determined by considering the Mellin transform of the spatial correlation of the velocity field

$$\tilde{D}^{\alpha\beta}(\mathbf{x}; m, z) = D_0 \xi \int_0^\infty \frac{dw}{w} \frac{1}{w^z} \int_{q>m} \frac{d^d q}{(2\pi)^d} \frac{e^{\imath w \mathbf{q}\cdot\mathbf{x}}}{q^{d+\xi}} \Pi^{\alpha\beta}(\hat{\mathbf{q}}). \tag{18}$$

The integral is convergent for $\Re z < 0$ and can be performed explicitly. The result is (see appendix B for details):

$$\tilde{D}^{\alpha\beta}(\mathbf{x}; m, z) = D\xi c(z) \frac{m^{z-\xi} x^z}{z(z-\xi)} \mathcal{T}^{\alpha\beta}(\hat{\mathbf{x}}, z) \tag{19}$$

with D and $\mathcal{T}^{\alpha\beta}$ respectively defined by (12) and (15) whilst

$$c(z) := \frac{(d-1+z)}{(d-1)} \frac{\Gamma(\frac{d+2}{2})\Gamma(1-\frac{z}{2})}{2^z \Gamma\left(\frac{d+2+z}{2}\right)}, \quad c(0) = 1 \tag{20}$$

is a function with simple poles for z a positive even integer.

The small scale asymptotics is derived by evaluating the inverse Mellin transform involving an integral over z along $\Re z = \text{const} < 0$ by pushing the contour to the right and picking residues from the poles. This gives

$$D^{\alpha\beta}(\mathbf{x}; m) = D m^{-\xi} \delta^{\alpha\beta} - d^{\star\alpha\beta}(\mathbf{x}) + o(m^{2-\xi} x^2). \tag{21}$$

Thus, the residue of the pole at zero corresponds to the eddy diffusivity. The pole at ξ specifies instead the inertial range asymptotic of the structure tensor of the velocity field:

$$d^{\star\alpha\beta}(\mathbf{x}) := D c(\xi) x^\xi \mathcal{T}^{\alpha\beta}\left(\hat{\mathbf{x}}, \xi\right). \tag{22}$$

Note that at ξ equal 2 the pole at z equal 2 in (19) turns from simple to double. This indicates the existence of logarithmic corrections to the analytic behavior, proportional to x^2, of the velocity field structure function. Logarithmic corrections are suppressed for example by redefining

$$D_0 \to \frac{D_0}{\Gamma(1-\frac{\xi}{2})}. \tag{23}$$

The rescaling does not affect universal quantities in the small ξ limit and will be neglected in the present paper.

Let us finally discuss the behavior of the velocity covariance around $\xi = 0$. At fixed ultra-violet cut-off it vanishes linearly with ξ. On the other hand, the removal of the ultra-violet cut-off reduces the dissipative range to the the single point x equal zero. There the velocity field coincides with the large scale constant mode. In the inertial range, by (21) the velocity field is vanishing with ξ for any nonzero m and x also after sending M to infinity. We will see that the expansion around ξ equal zero provides a viable analytic tool for the investigation of universal properties of advection.

4. HOPF'S EQUATIONS AND STATISTICAL CONSERVATION LAWS

Let $\theta(\mathbf{x}, t)$ be the solution of the stochastic differential Eq. (1) with suitable initial condition. The equal time correlation functions:

$$\mathcal{C}_{2n} := \prec \Pi_{i=1}^{2n} \theta(\mathbf{x}_i, t) \succ \tag{24}$$

satisfy in the Kraichnan model a solvable hierarchy of Hopf equations which are simplest to derive in the Ito representation[(22,47)] of the equation. Letting $\mathbf{v} \cdot \partial_{\mathbf{x}} \theta$ be defined with the Ito convention the Eq. (1) becomes

$$\partial_t \theta + \mathbf{v} \cdot \partial_{\mathbf{x}} \theta - \frac{1}{2}[\kappa \delta^{\alpha\beta} + D^{\alpha\beta}(0; m, M)] \partial_\alpha \partial_\beta \theta = f. \tag{25}$$

By (11), in the Ito representation the molecular viscosity is renormalized by the velocity field to

$$\varkappa = \kappa + D(m^{-\xi} - M^{-\xi}). \tag{26}$$

In a turbulent fluid macroscopic diffusion and mixing are dominated by the eddy diffusivity generated by the fluid motion. It is therefore physically justified to assume that the overwhelming contribution to (26) comes from the eddy diffusivity $\varkappa_\star$ introduced in equation (17).

A direct application of the Ito formula then yields

$$\partial_t \mathcal{C}_{2n} - \frac{\varkappa}{2} \sum_i \partial^2_{\mathbf{x}_i} \mathcal{C}_{2n} - \sum_{i<j} D^{\alpha\beta}(\mathbf{x}_i - \mathbf{x}_j; m, M) \partial_{\mathbf{x}_i^\alpha} \partial_{\mathbf{x}_j^\beta} \mathcal{C}_{2n} = \mathcal{F}_{2n} \tag{27}$$

with

$$\mathcal{F}_{2n} := \mathcal{C}_{(2n-2)} \otimes F \equiv \sum_{i<j} \mathcal{C}_{(2n-2)} (\mathbf{x}_1, \underset{\hat{i}\,\hat{j}}{\ldots}, \mathbf{x}_{2n}, t)\, F\left(\frac{\mathbf{x}_i - \mathbf{x}_j}{L_F}\right) \tag{28a}$$

$$\mathcal{F}_2 := F\left(\frac{\mathbf{x}_i - \mathbf{x}_j}{L_F}\right). \tag{28b}$$

Odd order correlation functions will be ignored since they vanish in the steady state due to parity invariance.

For a translation invariant initial condition for θ (say 0) $\mathcal{C}_{2n}$ is translation invariant i.e.

$$\sum_i \partial_{\mathbf{x}_i^\alpha} \mathcal{C}_{2n} = 0 \tag{29}$$

the Hopf equations reduce to the final form

$$\partial_t \mathcal{C}_{2n} - \frac{\kappa}{2} \sum_i \partial^2_{\mathbf{x}_i} \mathcal{C}_{2n} + \sum_{i<j} d^{\alpha\beta}(\mathbf{x}_i - \mathbf{x}_j; m, M) \partial_{\mathbf{x}_i^\alpha} \partial_{\mathbf{x}_j^\beta} \mathcal{C}_{2n} = \mathcal{F}_{2n}. \tag{30}$$

Note that owing to translational invariance the $2n$-point correlation in d dimensions is a function of $d_n = (2n-1)d$ variables.

Since (30) depends on the velocity statistics only through the structure function it has coefficients that have limits as the ultra-violet and infra-red cut-offs of the velocity field are removed. In that limit we may define the elliptic negative definite operators

$$\mathcal{M}^{\star}_{2n} := \sum_{i<j} d^{\star\alpha\beta}(\mathbf{x}_i - \mathbf{x}_j)\partial_{\mathbf{x}_i^\alpha}\partial_{\mathbf{x}_j^\beta}. \tag{31}$$

Since the coefficients $d^{\star}_{\alpha\beta}$ are homogeneous functions of degree ξ these differential operators are self similar, namely homogeneous of degree $\xi - 2$. The stationary state correlation functions of the scalar satisfy the equations

$$-\mathcal{M}^{\star}_{2n}\mathcal{C}_{2n} = \frac{\kappa}{2}\sum_{i=1}^{2n}\partial^2_{\mathbf{x}_i}\mathcal{C}_{2n} + \mathcal{F}_{2n}. \tag{32}$$

The terms collected on the right hand side of (32) are related to the non-universal dependencies of the dynamics. Self-similarity breaking from the large scale can occur through the integral scale of the forcing L_F. Comparing the and the $\mathcal{M}^{\star}_{2n}$ terms hints at the existence of a scalar field dissipative scale

$$\ell = \left(\frac{2\kappa}{D}\right)^{\frac{1}{\xi}}. \tag{33}$$

The existence and uniqueness of the solutions of (32) has been proved at zero molecular dissipation for all values of the Hölder exponent $\xi < 2$ of the velocity field.[37] They are given as

$$\mathcal{C}_{2n}(\mathbf{X}; L_F) = -\int d\mathbf{Y} M^{-1}_{2n}(\mathbf{X},\mathbf{Y})\mathcal{F}_{2n-2}(\mathbf{Y}; L_F) \tag{34}$$

for $\mathbf{X}, \mathbf{Y} \in \mathbb{R}^{d_n}$. The kernel M^{-1}_{2n} of the operator $\mathcal{M}^{\star -1}_{2n}$ was proved to be locally integrable and the integrals converge absolutely as long as the forcing scale L_F is finite. By the homogeneity of $\mathcal{M}^{\star}_{2n}{}^{-1}$ we get immediately that

$$\mathcal{C}_{2n}(\mathbf{X}, L_F) = L_F^{n(2-\xi)}\mathcal{C}_{2n}\left(\frac{\mathbf{X}}{L_F}, 1\right) \tag{35}$$

i.e. the canonical dimension of $\mathcal{C}_{2n}$ is $n(2-\xi)$.

5. ZERO MODES AND SHORT DISTANCE ASYMPTOTICS

From (35) we see that the large L_F behavior of the correlation functions $\mathcal{C}_{2n}$ is dominated by the large scale velocity: they blow up like $L_F^{n(2-\xi)}\mathcal{C}_{2n}(0, 1)$.

Sub-leading terms in the inertial range can be extracted by considering the Mellin transforms

$$\tilde{\mathcal{C}}_{2n}(\mathbf{X}, L_F; z) := \int_0^\infty \frac{dw}{w} \frac{\mathcal{C}_{2n}(w\mathbf{X}, L_F)}{w^z} = L_F^{n(2-\xi)} \frac{X^z}{L_F^z} \int_0^\infty \frac{dw}{w} \frac{\mathcal{C}_{2n}(w\hat{\mathbf{X}}, 1)}{w^z}. \tag{36}$$

These integrals converge for the real part of z small enough and are expected to extend to meromorphic functions of z at least for generic ξ.

The Mellin transform of the Hopf Eq. (32) is

$$\begin{aligned} &-\mathcal{M}^\star_{2n} X^{z+2-\xi} \tilde{\mathcal{C}}_{2n}(\hat{\mathbf{X}}, L_F; z+2-\xi) \\ &= \frac{\kappa}{2} \partial_X^2 X^{z+2} \tilde{\mathcal{C}}_{2n}(\hat{\mathbf{X}}, L_F; z+2) + X^z \tilde{\mathcal{F}}_{2n}(\hat{\mathbf{X}}, L_F; z). \end{aligned} \tag{37}$$

It was observed in Ref. 11 that poles of $\tilde{\mathcal{C}}_{2n}$ can occur either for values of z for which $\tilde{\mathcal{F}}_{2n}$ has a pole or for z such that the operator $\mathcal{M}^\star_{2n}$ has a zero mode

$$\mathcal{M}^\star_{2n} \mathfrak{z} = 0 \tag{38}$$

which is a homogeneous function of degree z. The poles of $\tilde{\mathcal{F}}_{2n}$ are in view of (28a) determined by solving the Hopf equations lower in the hierarchy. One then ends up with an asymptotic short distance expansion for $\mathcal{C}_{2n}$ in terms of homogeneous functions of the coordinates

$$\mathcal{C}_{2n}(\mathbf{X}, L_F) = \sum_j X^{\zeta_{n,j}} L_F^{n(2-\xi)-\zeta_{n,j}} A_j(\hat{\mathbf{X}}). \tag{39}$$

Since the forcing covariance is smooth and has a Taylor expansion $F(x) = \sum f_n |x|^n$ we conclude that the scaling exponents ζ_{nj} that may enter in (39) are either the homogeneity degrees of the zero modes of $\mathcal{M}_{2n}$ or they are determined in terms of the previous ones $\zeta_{n-1,j}$.

The general situation is illustrated by the case of the two-point function $\mathcal{C}_2$ which may be computed explicitly by quadrature if the forcing is isotropic[11]:

$$\mathcal{C}_2(x; L_F) = (d+\xi) \int_x^\infty dx_1 \frac{\int_0^{x_1} dx_2 F(x_2/L_F) x_2^{d-1}}{x_1^{d-1}[d^{\star\alpha}{}_\alpha(x_1) + (d+\xi)\kappa]} \tag{40}$$

with $d^{\star\alpha\beta}$ specified by (22). In such a case, at zero molecular viscosity, Eq. (37) reduces to

$$-d^{\star\alpha\beta}(\mathbf{x}) \partial_\alpha \partial_\beta x^{z+2-\xi} \tilde{\mathcal{C}}_2(1, L_F; z+2-\xi) = x^z \tilde{F}(1, L_F; z) \tag{41}$$

with solution

$$\tilde{\mathcal{C}}_2(x, L_F; z) = -L_F^{2-\xi} \frac{x^z}{L_F^z} \frac{(d+\xi-1)\tilde{F}(1, 1; z-2+\xi)}{c(\xi)(d-1)(d+z-2+\xi) z D}. \tag{42}$$

Zero modes correspond to the poles in z equal zero and $2-d-\xi$ respectively associated to short and large distance asymptotics.[18,24] The only zero mode contributing in short distances is the constant, the poles of $\tilde{F}$are at $z=2-\xi+n$ for nonnegative integer n.

6. ANOMALOUS SCALING OF THE STRUCTURE FUNCTIONS

Correlation functions that probe the sub-leading terms in the short distance expansion (39) are provided by the *structure functions* of the scalar namely

$$\mathcal{S}_{2n}(\mathbf{x}, L_F, \ell) := \prec [\theta(\mathbf{x}) - \theta(0)]^{2n} \succ \tag{43}$$

where we denoted explicitly the dependence on the forcing scale and κ via (33). Let us consider for simplicity isotropic forcing so that $\mathcal{S}_{2n}$ is only a function of $x=|\mathbf{x}|$. We may scale it out as

$$\mathcal{S}_{2n}(x, L_F, \ell) = x^{(2-\xi)n}\mathcal{S}_{2n}\left(1, \frac{L_F}{x}, \frac{\ell}{x}\right). \tag{44}$$

Suppose $\mathcal{S}_{2n}(1, L, \ell)$ had a limit as ℓ. tends to zero and L tends to infinity. Then we would conclude

$$\mathcal{S}_{2n}(x, L_F, \ell) = A_n x^{(2-\xi)n}\left(1 + o\left(\frac{x}{L_F}, \frac{\ell}{x}\right)\right), \tag{45}$$

i.e. the structure function scaling exponent would be $n(2-\xi)$. This is the prediction of the Obukhov–Corssin theory,[20,46] analogous to the Kolmogorov 41 theory[27,33,34,42] for Navier–Stokes turbulence that predicts there a scaling exponent $n/3$ for the velocity n-point structure function.

To discuss the validity of the Obukhov Corssin theory, we note first that the limit $\ell \to 0$ exists for the correlation functions and thus for the structure functions by the results of Ref. 37. The large L_F behavior depends on the nature of the terms entering the expansion (39), i.e. the zero modes of the inertial operators. Structure functions are obtained from correlation functions by applying the finite increment operator $\mathcal{I}_{\mathbf{x}}$

$$\mathcal{S}_{2n}(\mathbf{x}, t) = \mathcal{I}_{\mathbf{x}}\mathcal{C}_{2n}(\mathbf{x}_1, \ldots, \mathbf{x}_{2n}, t) \tag{46}$$

The operator $\mathcal{I}_{\mathbf{x}} = \Pi_i \iota_{\mathbf{x}}^{(i)}$ with i counting the number of fields in the correlation function generates finite field increments according to the rule $\iota_{\mathbf{x}} f(\mathbf{y}) = f(\mathbf{x}) - f(0)$. From (46) it follows immediately that the only zero modes of $\mathcal{C}_{2n}$ contributing to $\mathcal{S}_{2n}$ can be the *irreducible* ones, i.e. those depending on all the independent variables $\mathbf{x}_1 - \mathbf{x}_i$, $i = 2 \ldots n$.

The existence and the properties of zero modes were thoroughly investigated in Refs. 10, 11, 17, 18, 24, 30. It was shown in Refs. 10, 30 for small ξ and in Ref. 17 for large d that for each $n > 1$ there is a unique irreducible zero mode

$\mathfrak{Z}_{2n}$. The irreducible zero mode has scaling dimension $\zeta_{2n} = (2 - \xi)n - \rho_{2n}$ with $\rho_{2n} > 0$. In Ref. 11 further arguments were presented for the conclusion that this zero mode enters the expansion (39) and is the dominant term. This means that we may write

$$\mathcal{S}_{2n}(x, L_F, \ell) = x^{(2-\xi)n} \left(\frac{L_F}{x}\right)^{\rho_{2n}} s_{2n}\left(\frac{x}{L_F}, \frac{\ell}{x}, \xi\right) \left(\frac{F_\star}{D}\right)^n \tag{47}$$

where the function s_{2n} has a nonzero limit as $\ell \to 0$ and $L_F \to \infty$:

$$\lim_{L_F \to \infty} \lim_{\ell \to 0} s_{2n}\left(\frac{x}{L_F}, \frac{\ell}{x}, \xi\right) := s_{2n}(\xi) > 0. \tag{48}$$

The ratio between the constants $F_\star$ and D, defined respectively in (3) and (11), is introduced for later convenience.

It was also argued in Refs. 10, 30 that the sub-leading terms for the asymptotics of $\mathcal{S}_{2n}$ have exponents well separated from the leading one for small ξ, namely

$$s_{2n}\left(\frac{x}{L_F}, 0, \xi\right) := s_{2n}(\xi) + O\left(\left(\frac{x}{L_F}\right)^{2-O(\xi)}\right). \tag{49}$$

We may call this the *scaling Ansatz* for the Kraichnan model.

It is useful to express this result in terms of the Mellin transform. Let us shift for convenience z by $2n$ which is the ξ equal zero theory scaling exponent:

$$\tilde{\mathcal{S}}_{2n}(x, L; z + 2n) := \int_0^\infty \frac{dw}{w} \frac{\mathcal{S}_{2n}(wx, L)}{w^{xz+2n}} \tag{50}$$

which converges for $\Re z$ small enough (actually $\Re z < -O(\xi)$). No specific assumption is made on the origin of the integral scale L appearing in (50). The scaling Ansatz (49) then yields

$$\tilde{\mathcal{S}}_{2n}(x, L; z) := -\frac{A(z, \xi) L^{-n\xi - z} x^{2n+z}}{z - \sigma_{2n}(\xi)} \tag{51a}$$

$$\sigma_{2n}(\xi) := -n\xi - \rho_{2n}(\xi) \tag{51b}$$

with the function A having poles for values of z differing from σ_{2n} by terms at least of the order $O(\xi^0)$. Furthermore, A takes on the first pole of the Mellin transform the value

$$A(\sigma_{2n}(\xi), \xi) = s_{2n}(\xi) \left(\frac{F_\star}{D}\right)^n. \tag{52}$$

In the following sections structure functions will be computed using perturbation theory around ξ equal zero. It is useful to observe that the small ξ expansion of the Mellin transform generates a Laurent series in z. The residues of the expansion have simple relations with the expansion in powers of ξ of the scaling exponent:

$$\frac{d}{d\xi}\ln\left\{\tilde{\mathcal{S}}_{2n}(x,L;z+2n)L^{n\xi}\right\} = \frac{\sigma_{2n}'(0)}{z} + O(z^0,\xi^0)$$
$$+\xi\left[\frac{(\sigma_{2n}')^2(0)}{z^2} + \frac{\sigma_{2n}''(0)}{z} + O(z^0)\right] + O(\xi^2). \tag{53}$$

Analogous relations hold true to all orders in ξ and also for anisotropic forcing. Thus, the Mellin transform permits to extract, systematically at any order in ξ, zero mode contributions. Furthermore, as it will be seen below, taking the Mellin transform of the Feynman diagrams generated by the Martin-Siggia-Rose formulation[41] of the Kraichnan model greatly simplifies the explicit evaluation of the corresponding integrals. The fact is well known in field theory as the Mellin transform has the effect to map a theory with massive propagators into a massless one.[48,59]

7. MARTIN-SIGGIA-ROSE FORMALISM

The perturbative analysis of the zero modes of the inertial operators $\mathcal{M}_{2n}^{\star}$ in the parameter ξ is based on the fact that the velocity structure function (22) becomes constant as ξ tends to zero and the operators become Laplacians. It can then be checked that the solutions of the Hopf equations are correlation functions of a Gaussian field θ. For small ξ the distribution of θ should be given by a perturbation of this Gaussian. It is straightforward to derive a perturbation expansion from the Ito stochastic differential Eq. (25).

For this, let us observe that the solution of (25) can be written as

$$\theta(\mathbf{x},t) = \int d^dy\,\Re(\mathbf{x},t|\mathbf{y},t_0)\theta(\mathbf{y},t_0) + \int_{t_0}^{t} ds\int d^dy\,\Re(\mathbf{x},t|\mathbf{y},s)f(\mathbf{y},s) \tag{54}$$

with $\Re$ solution of the stochastic differential equation

$$\left(\partial_t - \frac{\varkappa}{2}\partial_{\mathbf{x}}^2\right)\Re = -\mathbf{v}\cdot\partial_{\mathbf{x}}\Re \tag{55}$$

with $\Re(\mathbf{x},t|\mathbf{y},t) = \delta^{(d)}(\mathbf{x}-\mathbf{y})$ and the product is defined with Ito convention. This equation can be solved as a series in multiple stochastic integrals

$$\Re(t|s) = \sum_0^{\infty}\int_s^{t_1}\dots\int_{t_n}^{t} R(t|t_n)\mathbf{v}(t_n)dt_n\cdot\partial R(t_n|t_{n-1})\dots\mathbf{v}(t_1)dt_1\cdot\partial R(t_1|s) \tag{56}$$

which converges in $\mathbb{L}^2$ of the probability space of $\mathbf{v}$.[39] Here R is the fundamental solution to the heat equation

$$R(\mathbf{x}, t|\mathbf{x}', t') = \int \frac{d^d p}{(2\pi)^d} e^{\imath \mathbf{p}\cdot(\mathbf{x}-\mathbf{x}') - \frac{\varkappa}{2} p^2 (t-t')}. \tag{57}$$

Inserting (56) to (54) and averaging over $\mathbf{v}$ and f leads to an expansion of the correlation functions in terms of integrals involving R and the covariances of $\mathbf{v}$ and f.

A convenient way to generate this series is provided by the so called Martin-Siggia-Rose (MSR) formalism (see Refs. 15, 51, 59, and references therein). It provides a graphical representation of the perturbation theory analogous to the Feynman rules of quantum field theory.

The MSR formalism is derived by introducing the generating function

$$\mathcal{Z}(\jmath, \bar{\jmath}) = \prec \int \mathcal{D}[\theta] e^{\langle \jmath, \theta \rangle} \delta(\partial_t \theta + \mathbf{v}\cdot\partial_{\mathbf{x}}\theta - \frac{\varkappa}{2}\partial^2_{\mathbf{x}}\theta - f - \imath\bar{\jmath}) \succ_{v,f} \tag{58}$$

where

$$\langle \jmath, \theta \rangle = \int_{-\infty}^{\infty} dt \int d^d x \, \jmath \theta \tag{59}$$

and $\prec \bullet \succ_{v,f}$ denotes the average with respect to the velocity and forcing fields. By introducing of an auxiliary "*ghost*" field $\bar{\theta}$ the MSR functional becomes

$$\mathcal{Z}(\jmath, \bar{\jmath}) = \prec \int \mathcal{D}[\theta\bar{\theta}] e^{\langle \jmath, \theta \rangle + \langle \bar{\theta}, \bar{\jmath} \rangle - \mathcal{A}} \succ_{f,v} \tag{60}$$

with

$$\mathcal{A} = -\imath \left\langle \bar{\theta}, \left(\partial_t + \mathbf{v}\cdot\partial - \frac{\varkappa\partial^2}{2} \right) \theta \right\rangle - \imath \langle \bar{\theta}, f \rangle. \tag{61}$$

Inspection of (58) evinces that functional derivatives of Z with respect to the source fields $\jmath, \bar{\jmath}$ evaluated in the origin generate averages of products of the scalar field θ and of its *response* to a variation of the force field. Inverting the order of integration and averaging over the velocity and forcing fields leads to

$$\mathcal{Z}(\jmath, \bar{\jmath}) = \prec e^{-\mathcal{A}_1(\theta,\bar{\theta}) - \frac{1}{2}\langle \bar{\theta}, F\bar{\theta} \rangle + \langle \jmath, \theta \rangle + \langle \bar{\theta}, \bar{\jmath} \rangle} \succ_G \tag{62}$$

where

$$\mathcal{A}_1(\theta, \bar{\theta}) = \frac{1}{2} \langle \bar{\theta}\partial_\alpha \theta, D^{\alpha\beta} \bar{\theta} \partial_\beta \theta \rangle \tag{63}$$

and the average is with respect the Gaussian "measure" with covariance given by

$$\prec \theta(\mathbf{x}, t)\theta(\mathbf{x}', t') \succ_G = \int \frac{d^d p}{(2\pi)^d} \frac{e^{\imath \mathbf{p}\cdot(\mathbf{x}-\mathbf{x}') - \frac{\varkappa}{2} p^2 |t-t'|}}{\varkappa p^2} F(\mathbf{p}) \tag{64}$$

and

$$\prec \theta(\mathbf{x}, t)\bar{\theta}(\mathbf{x}', t') \succ_G = \imath H_0(t - t')R(\mathbf{x}, t|x', t'). \tag{65}$$

Equation (65) defines the *response function* of the "free"-theory which is obtained by setting to zero the velocity field in the Ito representation of (1). It involves the "Ito-Heaviside function"

$$H_0(t) = \begin{cases} 1 & t > 0 \\ 0 & t \leq 0 \end{cases} \tag{66}$$

which insures that equal time contractions in (63) will not occur as follows from the Ito convention used for the stochastic integrals in (56) . The generating function satisfies the normalization

$$\mathcal{Z}(0, 0) = 1. \tag{67}$$

The perturbation expansion for $\mathcal{Z}(\jmath, \bar{\jmath})$ is obtained by expanding $e^{-\mathcal{A}_1}$ in powers of $\mathcal{A}_1$ and expressing the resulting expectations of θ, $\bar{\theta}$ in graphical terms as explained in Sec. 8. Before going to that let us make two comments concerning the infrared cutoffs and the small parameter in the expansion.

The free response function (65) depends on the eddy diffusivity (17) which diverges as the infrared cutoff m of the velocity field tends to zero. Thus this limit cannot be taken directly in (62). In Sec. 4 the properties of equal time scalar correlations were discussed in the case when $m = 0$. This limit can be taken in the MSR formalism only for equal time correlation and structure functions. In particular it can be shown[45] that order by order in perturbation theory, if the integral scale of the forcing L_F is kept fixed, the limit for m tending to zero exists and leads to the scaling predictions of Ref. 10.

On the other hand, in the MSR formalism it is natural to study the opposite limit when the integral scale of the velocity field m^{-1} is smaller than that of the forcing. As argued in Sec. 6, anomalous scaling of the structure functions is a consequence of the existence of homogeneous zero modes of Eq. (38) where the infrared cutoffs don't occur. Thus, it is plausible that the scaling exponents don't depend on the order in which the infrared cutoffs are removed. It would be interesting to spell this out more explicitly. In what follows we will keep m fixed and study the $m_F \to 0$ limit ($m_F \equiv L_F^{-1}$).

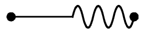
Finally let us comment on the small parameter of the expansion. In Sec. 3 it was shown that the spatial part of the velocity covariance vanishes almost everywhere (i.e. at $x \neq 0$) as $\xi \to 0$ whilst being bounded from above by the eddy diffusivity $\varkappa_\star$. Thus apart from the constant mode the (63) can be viewed as a small perturbation in the limit ξ tending to zero.

8. SMALL ξ EXPANSION

The general features of the small ξ expansion can be summarized as follows. The coefficients of the expansion are determined by integrals symbolically represented by Feynman diagrams. The basic ingredients of the Feynman diagrams are the free scalar correlation (also referred to as $\theta - \theta$-line) (64) and the free response function ($\theta - \bar{\theta}$-line) (65) together with the velocity correlation ($v - v$-line) (4). They have the graphical

$$\prec \theta(\mathbf{x}, t)\theta(\mathbf{x}', t') \succ_G = \quad (68a)$$

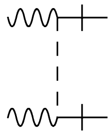
$$\prec \theta(\mathbf{x}, t)\bar{\theta}(\mathbf{x}', t') \succ_G = \quad (68b)$$



$$\prec v^{\alpha}(\mathbf{x}, t)v^{\beta}(\mathbf{x}', t') \succ = \quad (68c)$$

where end-line dots represent external points. In order to exhibit the expansion parameter, Feynman diagrams are constructed by connecting free response and correlations lines through the $O(\xi^0)$ part of the interaction vertex

$$\mathcal{A}_I = \frac{1}{\xi}\mathcal{A}_1 = \quad . \quad (69)$$



The bars transversal to scalar lines in (69) represent spatial derivatives. Finally, all contributions are reordered in powers of the Holder exponent of the velocity field by expanding the residual ξ dependence of the interaction.

It is worth stressing that this perturbation series is *ultraviolet finite* i.e. the resulting integrals have a well defined limit as the ultra-violet cut-off M tends to infinity. Singular behavior is instead exhibited in the limit of infinite integral scale, $m$tending to zero.

Structure functions are obtained by applying the finite increment operator (46) to the perturbative expressions of equal time scalar correlation functions. This operation removes order by order in perturbation theory all terms proportional to powers of the integral scale otherwise present for dimensional reasons in the correlation functions. The resulting perturbative expansion of structure functions contains only expressions exhibiting logarithmic divergences in the infra-red.

In Sec. 6 it was argued that $\mathcal{S}_{2n}$ is in the inertial range a homogeneous function of the spatial separation of degree ζ_{2n} determined by the unique irreducible zero-mode of the Hopf equation of order $2n$. According to (53) ζ_{2n} can be straightforwardly evaluated order by order in ξ by taking the Mellin transform of the perturbative expression of $\mathcal{S}_{2n}$. Working under these assumptions, in the following two subsections the calculation of $2n$ is outlined for the first two orders in ξ. For simplicity we set $M = \infty$ in the calculation. More details are deferred to appendices C and D.

8.1. Zeroth Order Approximation

The zeroth order of the perturbative expansion corresponds to neglecting the interaction term (63) in (62). In order to simplify the notation we keep the explicit non perturbative ξ dependence (17) in the eddy diffusivity $\varkappa_\star$. This trivial ξ dependence can be always expanded a posteriori to check on the decoupling of the scaling exponents into the part predicted by canonical dimensional analysis and a part associated to self-similarity breaking in the inertial range.

The second order structure function of the Gaussian theory is obtained from (64) and (2). In the limit $L_F \to \infty$ we get

$$\mathcal{S}_2^{(0)}(\mathbf{x}) = \lim_{L_F \uparrow \infty} \lim_{\kappa \downarrow 0} 2 \int \frac{d^d p}{(2\pi)^d} \frac{1 - e^{\imath \mathbf{p}\cdot\mathbf{x}}}{D p^2} L_F^d \check{F}(L_F p) = \frac{2}{D} \int \frac{d^d p}{(2\pi)^d} (\hat{\mathbf{p}} \cdot \mathbf{x})^2 \check{F}(p). \tag{70}$$

In particular, if the forcing correlation is isotropic we get

$$\mathcal{S}_2^{(0)}(\mathbf{x}) = \frac{x^2 F_\star}{d D}. \tag{71}$$

Structure functions of higher order are given in the Gaussian limit as

$$\mathcal{S}_{2n}^{(0)}(\mathbf{x}) = \frac{(2n)!}{2^n n!} \left[\mathcal{S}_2^{(0)}(\mathbf{x})\right]^n \tag{72}$$

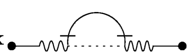
If the forcing is anisotropic, scaling properties of the structure functions are identified by expanding them in a functional basis invariant under rotations. In d-dimensions, this scope is achieved by resorting to an expansion in hyper-spherical harmonics.[54] Under rather general conditions, the generic Gaussian structure function takes the form

$$\mathcal{S}_{2n}^{(0)}(\mathbf{x}) = \left[\frac{x^2 F_\star}{d D}\right]^n \sum_j K_j Y_{j0}(\hat{\mathbf{x}}) + O\left(L_f^{-2}\right) \tag{73}$$

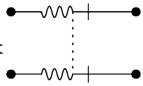
with K_j some non-universal constants depending on the forcing.

8.2. First Order Approximation

To first order in ξ the *structure functions* require the evaluation of the diagrams

$\mathcal{V}_{(1;2)} = \mathcal{I}_{\mathbf{x}}$  (74)

and

$$\mathcal{V}_{(1;4)} = \mathcal{I}_{\mathbf{x}}$$  . (75)

In (74), (75) the increment operator is given by Eq. (46). As $L_F \to \infty$, momenta flowing along $\theta - \theta$ lines tend to zero. In such a limit the two diagrams can be recast in the form

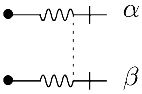
$$\mathcal{V}^{(0)}_{(1;2)} = \mathcal{V}^{(0)\alpha}_{(1;4)\alpha}\partial^2\mathcal{S}^{(0)}_2 \tag{76a}$$

$$\mathcal{V}^{(0)}_{(1;4)} = \mathcal{V}^{(0)\alpha\beta}_{(1;4)}\left(\partial_\alpha\mathcal{S}^{(0)}_2\right)\left(\partial_\beta\mathcal{S}^{(0)}_2\right) \tag{76b}$$

using the general notation

$$\mathcal{V}^{(n)\bullet}_{\bullet} := \left.\frac{d^n}{d\xi^n}\right|_{\xi=0}\mathcal{V}^{\bullet}_{\bullet} \tag{77}$$

to count the number of derivatives with respect to ξ of a diagram and

$$\mathcal{V}^{\alpha\beta}_{(1;4)} = \mathcal{I}_{\mathbf{x}}$$  $$= \frac{2D_0 m^\xi}{D}\int\frac{d^d p}{(2\pi)^d}\frac{1-e^{\imath\mathbf{p}\cdot\mathbf{x}}}{p^2}\frac{\Pi^{\alpha\beta}(\hat{\mathbf{p}})}{p^{d+\xi}}\chi_{[m,\infty]}(p). \tag{78}$$

In (78) the convention is adopted to denote with truncated scalar correlation lines in the diagrammatic representation the absence of momentum transfer along those lines. The factor two in (78) stems from the action of the finite increment operator $\mathcal{I}_{\mathbf{x}}$ on the diagram. From (78) it is readily verified the expected the ultra-violet convergence of $\mathcal{V}^{(0)\alpha\beta}_{(1;4)}$ as well as its logarithmic divergence in the infra-red.

The sum of (76a) and (76b) weighted by combinatorial factors yields the inertial range asymptotics of structure functions

$$\begin{aligned}\mathcal{S}_{2n}(\mathbf{x},m) = \frac{(2n)!}{2^n n!}\Big\{&\left[\mathcal{S}^{(0)}_2(\mathbf{x})\right]^n + \frac{n\xi}{2}\left[\mathcal{S}^{(0)}_2(\mathbf{x})\right]^{n-1}\mathcal{V}^{(0)\alpha\beta}_{(1;4)}\partial_\alpha\partial_\beta\mathcal{S}^{(0)}_2(\mathbf{x})\\ &+\frac{\xi n(n-1)}{2}\left[\mathcal{S}^{(0)}_2(\mathbf{x})\right]^{n-2}\mathcal{V}^{(0)\alpha\beta}_{(1;4)}\left[\partial_\alpha\mathcal{S}^{(0)}_2(\mathbf{x})\right]\left[\partial_\beta\mathcal{S}^{(0)}_2(\mathbf{x})\right]\Big\}\\ &+O(\xi^2, L_F^{-2}).\end{aligned} \tag{79}$$

Some straightforward algebra permits to recast the result in a more compact form

$$\mathcal{S}_{2n}(\mathbf{x};m) = \left\{1 + \frac{\xi}{2}\mathcal{V}^{(0)\alpha\beta}_{(1;4)}\partial_\alpha\partial_\beta\right\}\mathcal{S}^{(0)}_{2n}(\mathbf{x}) + O(\xi^2, L_F^{-2}). \tag{80}$$

The Mellin transform defined as in (50) acts on the perturbative expression of structure function as

$$\widetilde{\left[\mathcal{S}_{2n} - \mathcal{S}^{(0)}_{2n}\right]}(\mathbf{x},m;z+2n) = \frac{\xi}{2}\mathcal{V}^{(0)\alpha\beta}_{(1;4)}(z+2)\partial_\alpha\partial_\beta\mathcal{S}^{(0)}_{2n}(\mathbf{x}) + O(\xi^2, L_F^{-2}) \tag{81}$$

where

$$\mathcal{V}^{(0)\alpha\beta}_{(1;4)}(z+2) = \frac{2D_0}{D}\int_0^\infty \frac{dw}{w}\frac{1}{w^{z+2}}\int_{p\geq m}\frac{d^d p}{(2\pi)^d}\frac{1-e^{\imath w\mathbf{p}\cdot\mathbf{x}}}{p^2}\frac{\Pi^{\alpha\beta}(\hat{\mathbf{p}})}{p^d}. \tag{82}$$

The explicit evaluation of this integral is performed in appendix C.

The relations

$$\begin{aligned} x^2\partial^2(x^{2n}Y_{jl}) &= [2n(2n+d-2) - j(j+d-2)]x^{2n}Y_{jl} \\ \mathbf{x}^\beta\cdot\partial_\beta(x^{2n}Y_{jl}) &= 2nx^{2n}Y_{jl} \end{aligned} \tag{83}$$

together with (53) yield the first order result for the leading scaling exponents for each angular component of the structure functions:

$$\zeta_{2n,j} = 2n - \left[\frac{n(d+2n)}{d+2} - \frac{(d+1)j(d+j-2)}{2(d+2)(d-1)}\right]\xi + O(\xi^2). \tag{84}$$

The anomalous part of the scaling exponent is identified by expanding the ξ dependence of the free structure function upon the eddy diffusivity $\varkappa_\star$:

$$\rho_{2n,j} = \left[\frac{2n(n-1)}{d+2} - \frac{(d+1)j(d+j-2)}{2(d+2)(d-1)}\right]\xi + O(\xi^2). \tag{85}$$

The result is in agreement with those given in Refs. 5, 7, 8, 10. Note that the isotropic exponent $j=0$ dominates inertial range scaling.

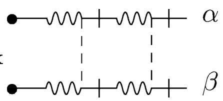
8.3. Second Order Approximation

To second order in ξ and as $L_F\to\infty$ we obtain the representation

$$\begin{aligned} \mathcal{S}_{2n}(\mathbf{x};m) = &\left\{1 + \frac{\xi}{2}\mathcal{V}^{(0)\alpha\beta}_{(1;4)}\partial_\alpha\partial_\beta + \frac{\xi^2}{2}\left[\mathcal{V}^{(1)\alpha\beta}_{(1;4)}\partial_\alpha\partial_\beta + \mathcal{V}^{(0)\alpha\beta}_{(2;4)}\partial_\alpha\partial_\beta\right]\right\}\mathcal{S}^{(0)}_{2n}(\mathbf{x}) \\ &+\xi^2\left\{\mathcal{V}^{(0)\alpha\beta\mu}_{(2;6)}\partial_\alpha\partial_\beta\partial_\mu + \frac{1}{8}\mathcal{V}^{(0)\alpha\beta\mu\nu}_{(2;8)}\partial_\alpha\partial_\beta\partial_\mu\partial_\nu\right\}\mathcal{S}^{(0)}_{2n}(\mathbf{x}) \\ &+ O(\xi^3, L_F^{-2}). \end{aligned} \tag{86}$$

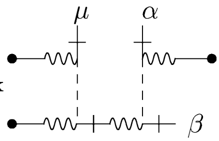
The expansion coefficients in (86) require the evaluation of two new diagrams

$\mathcal{V}^{\alpha\beta}_{(2;4)} = \mathcal{I}_{\mathbf{x}}$



$$= \frac{2D_0^2}{D^2}\int_{\substack{q\geq m\\ p\geq m}}\frac{d^dpd^dq}{(2\pi)^{2d}}\frac{1-\cos[(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}]}{(\mathbf{q}+\mathbf{p})^2}\frac{\mathbf{q}_\mu\mathbf{q}_\nu}{q^2}\frac{\Pi^{\mu\nu}(\hat{\mathbf{p}})\Pi^{\alpha\beta}(\hat{\mathbf{q}})}{p^{d+\xi}q^{d+\xi}} \tag{87}$$

$$\mathcal{V}^{\alpha\beta;\mu}_{(2;6)} = \mathcal{I}_{\mathbf{x}}$$

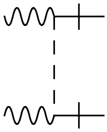


$$= \frac{-\imath D_0^2}{D^2} \int_{\substack{q \geq m \\ p \geq m}} \frac{d^d p d^d p}{(2\pi)^{2d}} \frac{(e^{\imath \mathbf{q}\cdot\mathbf{x}} - 1)(e^{-\imath(\mathbf{q}+\mathbf{p})\cdot\mathbf{x}} - 1)(e^{\imath \mathbf{p}\cdot\mathbf{x}} - 1)}{(q^2 + \mathbf{q}\cdot\mathbf{p} + p^2) q^2} \frac{\mathbf{q}_\nu \Pi^{\mu\nu}(\hat{\mathbf{p}}) \Pi^{\alpha\beta}(\hat{\mathbf{q}})}{p^{d+\xi} q^{d+\xi}}. \tag{88}$$

The diagram (88) is invariant under exchange of the indices α, β but not under exchange of μ with α or β. The fact is emphasized by the introduction of a *semicolon* separating the indices. In (86) index-contractions select only the *fully index-symmetric* part of (88). This is the reason why the *semicolons* between tensor indices appearing in (88) do not appear in (86).

The third new coefficient appearing in (86) for L_F tending to infinity is

$$\mathcal{V}^{\alpha\beta;\mu\nu}_{(2;8)} = \quad = \mathcal{V}^{\alpha\beta}_{(1;4)} \mathcal{V}^{\mu\nu}_{(1;4)} \tag{89}$$



Thus, the absence of momentum flow across inner scalar correlation lines reduces the evaluation of $\mathcal{V}^{(0)\alpha\beta;\mu\nu}_{(2;8)}$ and $\mathcal{V}^{(1)\alpha\beta}_{(1;4)}$ to the first order integral computed in appendix C.

As expected by dimensional considerations, the integrals are convergent for large momentum values and logarithmic for small momenta.

The Mellin transform (50) of (86) acts on the individual Feynman diagrams as

$$\begin{aligned}
&\widetilde{[\mathcal{S}_{2n} - \mathcal{S}^{(0)}_{2n}]}(\mathbf{x}, m; z + 2n) \\
&\quad = \frac{1}{2} \left\{ \partial \tilde{\mathcal{V}}^{(0)\alpha\beta}_{(1;4)}(z+2) + \xi^2 \left[\tilde{\mathcal{V}}^{(1)\alpha\beta}_{(1;4)}(z+2) + \tilde{\mathcal{V}}^{(0)\alpha\beta}_{(2;4)}(z+2) \right] \right\} \partial_\alpha \partial_\beta \mathcal{S}^{(0)}_{2n}(\mathbf{x}) \\
&\qquad + \xi^2 \left\{ \tilde{\mathcal{V}}^{(0)\alpha\beta\mu}_{(2;6)}(z+3) \partial_\alpha \partial_\beta \partial_\mu + \frac{1}{8} (\widetilde{\mathcal{V}^{(0)\alpha\beta}_{(1;4)} \mathcal{V}^{(0)\mu\nu}_{(1;4)}})(z+4) \partial_\alpha \partial_\beta \partial_\mu \partial_\nu \right\} \mathcal{S}^{(0)}_{2n}(\mathbf{x}) \\
&\qquad + O\big(\xi^3, L_F^{-2}\big)
\end{aligned} \tag{90}$$

where

$$\tilde{\mathcal{V}}^{(0)\alpha\beta}_{(2;4)}(z+2) = \int_0^\infty \frac{dw}{w} \frac{2D_0^2}{D^2 w^{z+2}} \int_{\substack{q \geq m \\ p \geq m}} \frac{d^d p d^d q}{(2\pi)^{2d}} \frac{1 - e^{\imath w \mathbf{p}\cdot\mathbf{x}}}{(\mathbf{q}+\mathbf{p})^2} \frac{\mathbf{q}_\mu \mathbf{q}_\nu}{q^2} \frac{\Pi^{\mu\nu}(\hat{\mathbf{p}}) \Pi^{\alpha\beta}(\hat{\mathbf{q}})}{p^d q^d} \tag{91}$$

$$\mathcal{V}^{(0)\alpha\beta;\mu}_{(2;6)}(z+3) = \int_0^\infty \frac{dw}{w} \frac{-\imath D_0^2}{D^2 w^{z+3}}$$

$$\times \int_{\substack{q \geq m \\ p \geq m}} \frac{d^d q d^d p}{(2\pi)^{2d}} \frac{(e^{\imath w \mathbf{q}\cdot\mathbf{x}} - 1)(e^{-\imath w(\mathbf{q}+\mathbf{p})\cdot\mathbf{x}} - 1)(e^{\imath w \mathbf{p}\cdot x} - 1)}{(q^2 + \mathbf{q}\cdot\mathbf{p} + p^2)q^2} \frac{\mathbf{q}_\nu \Pi^{\mu\nu}(\hat{\mathbf{p}})\Pi^{\alpha\beta}(\hat{\mathbf{q}})}{p^d q^d}. \tag{92}$$

Guidelines for the evaluation and explicit expressions of the Mellin transform of the diagrams are deferred to appendix D.

Representing (86) in terms of hyperspherical harmonics[54] yields the scaling exponents of the structure function for each value of the angular momentum j:

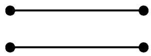
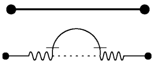
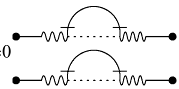
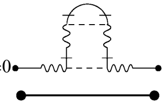
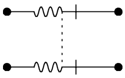
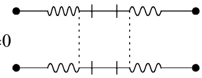
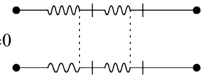
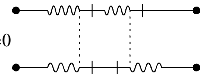
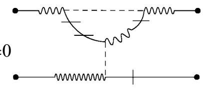
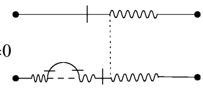
$$\zeta_{2n,j} = 2n - \left[\frac{n(d+2n)}{d+2} - \frac{(d+1)j(d+j-2)}{2(d+2)(d-1)}\right]\xi + (d+1)\left\{\frac{4(16-5d)n^2}{(d-1)(d+2)^3(d+4)}\right.$$
$$+ \frac{4[4-(j-2)j+d^2(5+2j)+dj(2j-5)-9d]n - 48(d-1)n^3 - 9dj(d+j-2)}{(d-1)^2(2+d)^3(4+d)}$$
$$+ (n-1)\mathrm{Hyp}_{21}\left(1, 1, 2+\frac{d}{2}, \frac{1}{4}\right)\left[\frac{3d(6+d)n}{(2+d)^3(4+d)(1+d)} + 6n\frac{n(4+d(7+d))-4}{(d^2-1)(2+d)^3(4+d)}\right.$$
$$\left.\left. - \frac{3(d^3+6d^2+d-4)j(d+j-2)}{2(d-1)^2(2+d)^3(4+d)(d+1)}\right]\right\}\xi^2 + O(\xi^3) \tag{93}$$

with Hyp_{21} denotes the Gauss hypergeometric series[1]:

$$\mathrm{Hyp}_{21}(a,b,c,x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^\infty \frac{\Gamma(a+n)\Gamma(b+n)x^n}{\Gamma(c+n)\Gamma(n+1)}. \tag{94}$$

The result is in agreement with those of Refs. 2, 3, 5 derived using the ultra-violet renormalization group. From the computational point of view the Mellin transform applied here to the perturbative expansion does not provide the simplest scheme to derive the scaling exponents. It is conceptually important because it shows how to relate zero modes of the Hopf equations to the more general diagrammatic expansion.

For completeness sake, it is worth noticing that gluing together truncated scalar correlation lines of (87), (88) and (89) permits for any given structure function to reconstruct the diagrammatic expression of the non-universal contributions vanishing in the limit L_F tending to infinity. For example, the full diagrammatic expression of the fourth order structure function is

$$\mathcal{S}_4(\mathbf{x}; m) = 3\,\mathcal{I}_{\mathbf{x}}$$

$$+\,6\left(\xi\mathcal{I}_{\mathbf{x}}|_{\xi=0} + \xi^2\mathcal{I}_{\mathbf{x}}\frac{d}{d\xi}\bigg|_{\xi=0}\right)$$

$$+\,3\xi^2\mathcal{I}_{\mathbf{x}}|_{\xi=0}$$

$$+\,3\xi^2\mathcal{I}_{\mathbf{x}}|_{\xi=0}$$

$$+\,12\left(\xi\mathcal{I}_{\mathbf{x}}|_{\xi=0} + \xi^2\mathcal{I}_{\mathbf{x}}\frac{d}{d\xi}\bigg|_{\xi=0}\right)$$

$$+\,6\xi^2\mathcal{I}_{\mathbf{x}}|_{\xi=0}$$

$$+\,12\xi^2\mathcal{I}_{\mathbf{x}}|_{\xi=0}$$

$$+\,24\xi^2\mathcal{I}_{\mathbf{x}}|_{\xi=0}$$

$$+\,24\xi^2\mathcal{I}_{\mathbf{x}}|_{\xi=0}$$

$$+\,24\xi^2\mathcal{I}_{\mathbf{x}}|_{\xi=0}$$

$$+\,0)(\xi^3). \tag{95}$$

9. GRADIENT CORRELATIONS AND THE ROLE OF THE DISSIPATIVE SCALE

From the short distance behavior of the structure functions of the scalar for zero molecular diffusivity we infer that the the field $\theta(\mathbf{x})$ will not be differentiable: it is only Holder continuous with exponent $1 - O(\xi)$. Hence correlation functions of the gradients of θ should blow up as the dissipative scale is taken to zero.

The scaling Ansatz (47) imposes the existence of a precise relation between the rate of these divergences and the scaling exponents of the structure functions. It is convenient to illustrate the argument in the fully isotropic case, the generalization to anisotropy being straightforward. Using the hypothesis of inertial range universality, L and ℓ will represent respectively the integral and dissipative scale of the scalar field disregarding of the mechanism responsible for their onset.

Radial gradient correlations at equal points and structure functions are related by the incremental ratio

$$\begin{aligned}\mathcal{G}_{2n}(L,\ell) &:= \prec \left[\tfrac{\mathbf{x}^\alpha}{x}\partial_\alpha\theta(\mathbf{x})\right]^{2n} \succ = \lim_{x\downarrow 0} \prec \left[\tfrac{\theta(\mathbf{x})-\theta(0)}{x}\right]^{2n} \succ. \\ &= \lim_{x\downarrow 0} x^{-2n} S_{2n}(x,\ell,L)\end{aligned} \tag{96}$$

By scaling

$$S_{2n}(x,\ell,L) = x^{2n}\ell^{-\xi n} c_n(L/\ell).$$

Supposing

$$c_n(L/\ell) \sim (L/\ell)^{\alpha_n} \tag{97}$$

and matching at $x = \ell$ with the scaling Ansatz (47) we infer

$$\mathcal{G}_{2n}(L,\ell) \propto \ell^{-n\xi} \left(\frac{L}{\ell}\right)^{\rho_{2n}}. \tag{98}$$

As the dissipative scale tends to zero gradient correlations are seen to blow up at equal points as a power law with exponent determined by the anomalous scaling exponent of the structure functions. Thus (98) relates the existence of anomalous scaling and the *dissipative anomaly* in the energy flux of the scalar field. Taking the angular average of (98) relates it to the dissipative anomaly

$$\mathcal{G}_{2n}(L,\ell) = \prec [(\partial^{\alpha}\theta\partial_{\alpha}\theta)(0,t)]^n \succ \int \frac{d\Omega_d}{\Omega_d} \cos^{2n}_{\angle}(\mathbf{x},\partial\theta) \tag{99}$$

where the symbol $\angle$ means that the argument of the cosine is the angle between $\mathbf{x}$ and the gradient of θ. The explicit expression of the angular average is given by

$$\int \frac{d\Omega_d}{\Omega_d} \cos^{2n}_{\angle}(\mathbf{x},\partial\theta) = \frac{\Gamma(2n+1)\Gamma(\frac{d}{2})}{4^n\Gamma(n+1)\Gamma(\frac{d}{2}+n)}. \tag{100}$$

9.1. Perturbative Expansion for Radial Gradient Correlations

Equation (98) suggests that it should be possible to determine the scaling exponents from a perturbative expansion of radial gradient correlations rather than of the structure functions.[8] By dimensional analysis a similar expansion is seen to generate Feynman diagrams logarithmic at all momentum scales. The evaluation of the associated integrals is therefore greatly simplified.

The identity

$$\prec (\mathbf{x}^{\alpha}\partial_{\alpha}\theta)^{2n} \succ = x^{2n} \prec (\hat{\mathbf{x}}^{\alpha}\partial_{\alpha}\theta)^{2n} \succ = x^{2n}\mathcal{G}_{2n} \tag{101}$$

permits to derive the perturbative expansion of radial gradients from

$$\begin{aligned}
&\prec (\mathbf{x}^{\alpha}\partial_{\alpha}\theta)^{2n} \succ \\
&= \left\{1 + \frac{\xi}{2}\mathcal{U}^{(0)\alpha\beta}_{(1;4)}\partial_{\alpha}\partial_{\beta} + \frac{\xi^2}{2}\left[\mathcal{U}^{(1)\alpha\beta}_{(1;4)}\partial_{\alpha}\partial_{\beta} + \mathcal{U}^{(0)\alpha\beta}_{(2;4)}\partial_{\alpha}\partial_{\beta}\right]\right\} \prec (\mathbf{x}^{\alpha}\partial_{\alpha}\theta)^{2n} \succ G \\
&\quad + \xi^2 \left\{\mathcal{U}^{(0)\alpha\beta\mu}_{(2;6)}\partial_{\alpha}\partial_{\beta}\partial_{\mu} + \frac{1}{8}\mathcal{U}^{(0)\alpha\beta}_{(1;4)}\mathcal{U}^{(0)\mu\nu}_{(1;4)}\partial_{\alpha}\partial_{\beta}\partial_{\mu}\partial_{\nu}\right\} \prec (\mathbf{x}^{\alpha}\partial_{\alpha}\theta)^{2n} \succ_G + O(\xi^3)
\end{aligned} \tag{102}$$

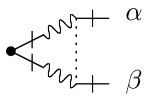
where $\prec \bullet \succ_G$ denotes averaging with respect to the Gaussian measure. The argument of the logarithms appearing in the expansion is now the ratio between the integral and dissipative scales of the velocity field. As in Sec. 8 the result is written by keeping fixed the eddy diffusivity (17) and in the limit of integral scale of the forcing L_F tending to infinity.

The $\mathcal{U}$-coefficients in (102) are related through the general definition

$$\mathcal{U}^{(n)\bullet}_{\bullet} = \frac{d^n}{d\xi^n}\bigg|_{\xi=0} \mathcal{U}^{\bullet}_{\bullet} \tag{103}$$

to the evaluation of the Feynman diagrams

$\mathcal{U}^{\alpha\beta}_{(1;4)} =$

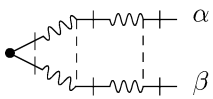


$$= \frac{D_0 m^{\xi}}{D} \int_{M \geqslant p \geqslant m} \frac{d^d p}{(2\pi)^d} \frac{(\mathbf{p}\cdot\mathbf{x})^2}{p^2} \frac{\Pi^{\alpha\beta}(\hat{\mathbf{p}})}{p^{d+\xi}}$$

$$= \frac{(d+1)x^2}{(d-1)(d+2)\xi}\left[1 - \left(\frac{M}{m}\right)^{-\xi}\right] \mathcal{T}^{\alpha\beta}(\hat{\mathbf{x}}, \mathbf{2}) \tag{104}$$

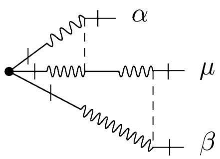
and

$\mathcal{U}^{\alpha\beta}_{(2;4)} =$



$$= \frac{D_0^2}{D^2} \int_{\substack{M \geq q \geq m \\ M \geq p \geq m}} \frac{d^d p d^d q}{(2\pi)^{2d}} \frac{[(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}]^2}{(\mathbf{q}+\mathbf{p})^2} \frac{\mathbf{q}_\mu \mathbf{q}_\nu}{q^2} \frac{\Pi^{\mu\nu}(\hat{\mathbf{p}})\Pi^{\alpha\beta}(\hat{\mathbf{q}})}{p^{d+\xi} q^{d+\xi}} \tag{105}$$

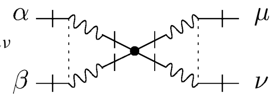
$\mathcal{U}^{\alpha\beta}_{2;6} =$



$$= \frac{D_0^2}{D^2} \int_{\substack{M \geq q \geq m \\ M \geq p \geq m}} \frac{d^d q d^d p}{(2\pi)^{2d}} \frac{(\mathbf{q}\cdot\mathbf{x})[(\mathbf{q}+\mathbf{p})\cdot\mathbf{x}](\mathbf{p}\cdot\mathbf{x})}{(q^2+\mathbf{q}\cdot\mathbf{p}+p^2)q^2} \frac{\mathbf{q}_\nu \Pi^{\mu\nu}(\hat{\mathbf{p}})\Pi^{\alpha\beta}(\hat{\mathbf{q}})}{p^{d+\xi} q^{d+\xi}}. \tag{106}$$

The evaluation of the last coefficient appearing in (102) does not require the evaluation of further integrals

$\mathcal{U}^{\alpha\beta;\mu\nu}_{(2;8)}$



$$= \mathcal{U}^{\alpha\beta}_{(1;4)} \mathcal{U}^{\mu\nu}_{(1;4)} \tag{107}$$

As in the case of structure function only the index-symmetric part of the diagrams above matters in the evaluation of (102) whence the omission of the semicolon between indices.

The Mellin transform proves again useful in extracting logarithms from (102). The transform is well defined since the condition $M > m$ provides an infra-red cut-off:

$$\tilde{\mathcal{G}}_{2n}(m, M; z) := \int_m^\infty \frac{dw}{w} \frac{\mathcal{G}_{2n}(Mw, m)}{w^z} \propto \frac{m^{n\xi}}{z - n\xi - \rho_{2n}} \left(\frac{M}{m}\right)^z . \tag{108}$$

Evaluated at m equal one or equivalently by keeping $\varkappa_\star$ constant in the ξ-expansion, (108) permits to apply the formula (53) to the computation of the scaling exponent. The explicit expressions of the Mellin transform of the $\mathcal{U}$-coefficients together with some details about their computation are given appendix F. Straightforward algebra then shows that from (102) the $O(\xi^2)$ result (93) for the anomalous scaling exponent is recovered. Finally it is readily verified that (93) can be related to the perturbative expansion of the structure function in the limit

$$\mathcal{G}_{2n}(m, M) = \frac{\prec (\mathbf{x}^\alpha \partial_\alpha \theta)^{2n} \succ}{x^{2n}} = \lim_{x \downarrow 0} \frac{\mathcal{S}_{2n}(x; m^{-1}, M^{-1})}{x^{2n}} . \tag{109}$$

Namely, the $\mathcal{U}$-coefficients in (93) coincide with leading order of the Taylor expansion in the spatial increments of the $\mathcal{V}$-coefficients in (86).

10. RENORMALIZATION GROUP METHODS AND THE VALIDATION OF SCALING ANSÄTZE

The zero mode analysis leading to the scaling Ansatz (47) justifies the exponentiation of the logarithms encountered in the perturbative expansion in powers of ξ. However, the concept of a zero mode is based on the existence of a closed hierarchy of Hopf equations for equal time correlations. This is a very special feature of the Kraichnan model which is not present in more realistic models of fluid turbulence. It is therefore useful to look for a validation of scaling directly in framework of the MSR formalism.

Such a validation is well known in the statistical field theory of dynamical and critical phenomena namely the Wilson formulation of the renormalization group (see for example Refs. 56, 57) and by its subsequent refinements (comprehensive presentations can be found for example in the monographs[(14,19,40,43,51,59)]).

In the theory of critical phenomena scaling of correlation functions occurs at large spatial scales. They exhibit at the critical point a power law fall-off which is insensitive to the short distance details of the system. The universality of the long distance or infra-red behavior with respect to the ultra-violet is formalized by direct or ultra-violet renormalization. In the coming subsection the main idea will be recalled with the aim of applying it to the Kraichnan model where universality is expected to emerge from the renormalization of *infra-red* rather than *ultra-violet* degrees of freedom once the dissipative scale has been set to zero.

10.1. Wilson Direct (Ultra-Violet) Renormalization Group

We start by briefly recalling the Wilson renormalization group as applied to the study of correlation functions at large spatial scales. A field theory or a spin system is specified by an action functional $\mathcal{A}$ (or Hamiltonian) depending on fields, spins and the like here collectively denoted by ϕ where ϕ is a random field $\phi(\mathrm{x})$ with probability distribution formally given by

$$\mathcal{P}[\phi] \propto e^{-A(\phi)}\mathcal{D}[\phi]. \tag{110}$$

Local observables $\mathcal{O}(x)$ are functions of ϕ and its derivatives at the point x. In analogy to quantum field theory, they are often called *operators.*

At the critical point correlation functions of local operators exhibit scaling

$$\prec \mathcal{O}(\mathbf{x})\mathcal{O}(\mathbf{y}) \succ \overset{|\mathbf{x}-\mathbf{y}|\uparrow\infty}{\sim} |\mathbf{x}-\mathbf{y}|^{2\eta_{\mathcal{O}}} \tag{111}$$

with $\eta_{\mathcal{O}}$ the *scaling dimension* of the operator $\mathcal{O}$. Scaling becomes exact in the *scaling limit* i.e. for the random fields

$$\mathcal{O}_{\star}(\mathbf{x}) = \lim_{\lambda\downarrow 0} \lambda^{-\eta_{\mathcal{O}}}\mathcal{O}(\lambda^{-1}\mathbf{x}) \tag{112}$$

whose two point function is

$$\prec \mathcal{O}_{\star}(\mathbf{x})\mathcal{O}_{\star}(\mathbf{y}) \succ \propto |\mathbf{x}-\mathbf{y}|^{-2\eta_{\mathcal{O}}}. \tag{113}$$

Thus, the scaling operator $\mathcal{O}_{\star}(\mathbf{x})$ describes the long distance behavior of $\mathcal{O}(\mathbf{x})$.

In statistical mechanics the fields ϕ and hence $\mathcal{O}$ have an UV cutoff (e.g. the lattice spacing). $O_{\star}$ has no such cutoff. Wilson's idea was to combine the scaling limit with a coarse graining operation so that the fields retain a fixed UV cutoff and the operation, called Renormalization Group, can be viewed as a mapping on probability distributions (or actions). This leads to a theory of the scaling dimensions $\eta_{\mathcal{O}}$.

In the simplest setup the fields ϕ have a cutoff in momentum space e.g. defined by having Fourier transform with support for radial values of the momentum in $[0, M]$. Let φ consist of the Fourier components of ϕ in the range $[0, \lambda M]$ and $\delta\phi$ the ones on $[\lambda M, M]$:

$$\phi(\mathbf{x}) = \varphi(\mathbf{x}) + \delta\phi(\mathbf{x}). \tag{114}$$

Define a rescaling operation

$$\mathcal{O}_{(\lambda)}(\mathbf{x}) := \lambda^{-\eta_{\mathcal{O}}}\mathcal{O}(\lambda^{-1}\mathbf{x}) \tag{115}$$

and let

$$\phi'(x) = \varphi_{(\lambda)}(\mathbf{x}) \tag{116}$$

so ϕ' has momenta on $[0, M]$. Then we have the decomposition

$$\phi(\mathbf{x}) = \phi'_{(1/\lambda)}(\mathbf{x}) + \delta\phi(\mathbf{x}). \tag{117}$$

The expectation of $\mathcal{O}$ can be rewritten as

$$\prec \mathcal{O} \succ = \frac{\int \mathcal{D}[\phi]\mathcal{O}(\phi)e^{-\mathcal{A}(\phi)}}{\int \mathcal{D}[\phi]e^{-\mathcal{A}(\phi)}} = \frac{\int \mathcal{D}[\varphi]\mathcal{D}[\delta\phi]\mathcal{O}(\phi'_{(1/\lambda)} + \delta\phi)e^{-\mathcal{A}(\phi'_{(1/\lambda)}+\delta\phi)}}{\int \mathcal{D}[\varphi]\mathcal{D}[\delta\phi]e^{-\mathcal{A}(\phi'_{(1/\lambda)}+\delta\phi)}}. \tag{118}$$

Integrating over $\delta\phi$ this becomes

$$\prec \mathcal{O} \succ = \frac{\int D[\phi'](\mathcal{L}_\lambda\mathcal{O})(\phi')e^{-(\mathcal{R}_\lambda\mathcal{A})(\phi')}}{\int \mathcal{D}[\phi']e^{-(\mathcal{R}_\lambda\mathcal{A})(\phi')}} \tag{119}$$

where we defined the coarse-grained or renormalized operator

$$(\mathcal{L}_\lambda\mathcal{O})(\phi') := \frac{\int \mathcal{D}[\delta\phi]\mathcal{O}(\phi'_{(1/\lambda)} + \delta\phi)e^{-\mathcal{A}(\phi'_{(1/\lambda)}+\delta\phi)}}{\int \mathcal{D}[\delta\phi]e^{-\mathcal{A}(\phi'_{(1/\lambda)}+\delta\phi)}} \tag{120}$$

and the renormalized action functional

$$(\mathcal{R}_\lambda\mathcal{A})(\phi') = -\ln\int \mathcal{D}[\delta\phi]e^{-\mathcal{A}(\phi'_{(1/\lambda)}+\delta\phi)}. \tag{121}$$

These functionals depend on the field ϕ which has momentum support on the original range $[0, M]$. $\mathcal{R}_\lambda$ is called the renormalization group transformation and clearly $\mathcal{L}_\lambda$ is the derivative $D\mathcal{R}_\lambda$ of $\mathcal{R}_\lambda$ at $\mathcal{A}$ i.e. the linearized RG transformation.

The renormalization group transformation $\mathcal{R}_\lambda$ satisfies the relation

$$\mathcal{R}_{\lambda_1}\mathcal{R}_{\lambda_2} = \mathcal{R}_{\lambda_1\lambda_2} \tag{122}$$

and defines a semi-group acting on the space of field functionals.

The physical interpretation of RG is that by *averaging* over the ultra-violet degrees of freedom stored in $\delta\phi$ and *rescaling* the resulting theory coincides in the long distances with the original theory and differs only in irrelevant short distance properties. The limit λ tending to zero should then describe universal long distance properties common to many action functionals $\mathcal{A}$. It plays the role of the scaling limit in the Wilson formulation. The simplest limit is a fixed point

$$\mathcal{A}_\star = \mathcal{R}_\lambda\mathcal{A}_\star \tag{123}$$

and given an action $\mathcal{A}$ there is at most one choice of $\eta\phi$ of the scaling dimension of the basic fields such that the renormalization group flow of the action $\mathcal{A}$ converges to a fixed point.

Scaling fields in the Wilson theory are local eigenoperators of the linearized RG $\mathcal{L}^*_\lambda$ at the fixed point $\mathcal{A}_\star$:

$$\mathcal{L}^\star_\lambda(\mathbf{x}) = \lambda^{\eta_\mathcal{O}} \mathcal{O}(\lambda \mathbf{x}). \tag{124}$$

These may often be found as follows. Suppose the limits

$$\lim_{\lambda \downarrow 0} \mathcal{R}_\lambda \mathcal{A} = \mathcal{A}_\star \tag{125}$$

and

$$\lim_{\lambda \downarrow 0} (\mathcal{L}_\lambda \mathcal{O})_{(\lambda)} = \mathcal{O}_\star. \tag{126}$$

exist. Then $\mathcal{O}_\star$ satisfies Eq. (124).

We formulated the UV renormalization as the search of universal long distance properties of a theory with fixed UV cutoff. In field theory one is also interested in the continuum limit, i.e. the removal of the UV cutoff, $M \to \infty$. Thus one considers a one parameter family of actions $\mathcal{A}^M$ with UV cutoff M and possibly depending parametrically on M (through bare mass, coupling constant, wave function renormalization etc). One then fixes some momentum scale $\bar{m}$ and splits the field as

$$\phi(\mathbf{x}) = \varphi(\mathbf{x}) + \delta\phi(\mathbf{x}) \tag{127}$$

where φ has UV cutoff $\bar{m}$ and the fluctuation part $\delta\phi$ momenta between $\bar{m}$ and M. Call the result after integrating over $\delta\phi$ by $e^{-\mathcal{A}^M_{\bar{m}}(\varphi)}$. The limit

$$\lim_{M \to \infty} \mathcal{A}^M_{\bar{m}} = \mathcal{A}_{\bar{m}} \tag{128}$$

is the effective action of scale $\bar{m}$. This problem of continuum limit is obviously related to the previous one by trivial rescalings. For the limits to exist, $\mathcal{A}^M$ (after rescaling to say unit cutoff) has to approach the stable manifold of a fixed point $\mathcal{A}_\star$ of the Wilson RG as $M \to \infty$ and then $\mathcal{A}_{\bar{m}}$ (after rescaling again) will lie on the unstable manifold of $\mathcal{A}_\star$.

The Wilson idea can be applied with small changes to the time dependent correlation functions of solutions of stochastic (partial) differential equations. In that case the fields $\phi(\mathbf{x}, t)$ depend also on time and P is given by the MSR construction (and is not positive). The coarse graining takes place in space only whereas in time one scales. We write

$$\mathcal{O}_{(\lambda)}(\mathbf{x}, t) := \lambda^{-\eta_\mathcal{O}} \mathcal{O}(\lambda^{-1}\mathbf{x}, \lambda^{\eta_t} t) \tag{129}$$

where the time exponent η_t has to be determined. In the simplest diffusion process $\eta_t = -2$.

10.2. Infra-red Renormalization

In the theory of critical phenomena the concept of universality refers to the independence of the long distance properties of correlation functions on the short distance details of the Hamiltonian which in the RG language translates to the fact that all the Hamiltonians in the domain of attraction of a given fixed point have the same critical exponents.

In turbulence there is an inversion of scales. There are many mechanisms which may stir the onset of turbulence at large scales and universality refers to the independence of the inertial range scaling on these long distance details of the forcing. It has therefore been conjectured in the literature (see Refs. 23, 32 for a general discussion and further references) that it should be possible to prove the irrelevance of infra-red degrees of freedom for inertial range scaling through the use of an inverse renormalization group. The adjective "inverse" must be understood in the sense that the renormalization group is constructed by averaging over fluctuations $\delta\phi$ with support in the infra red. More precisely, Wilson's recursion scheme is implemented along the same lines of the direct one encoded in formulae (117)–(123) but with the following differences

(i) The basic field ϕ is supported for radial values of the momentum in $[m, \infty]$ with m the infra-red cut-off. It is decomposed into a scaling field $\phi'_{(1/\lambda)}$ with support in $[\lambda m, \infty]$ and fluctuating field $\delta\phi$ with support in $[m, \lambda m]$. In view of the inversion, the asymptotic regime is reached now for large values of λ.

(ii) If the (inversely) renormalized actions $\mathcal{A}_\lambda$ converge to a fixed point $\mathcal{A}_\star$ as λ tends to infinity then $\mathcal{A}_\star$ describes the universal short distance properties of the theory.

In the following section the idea of inverse renormalization will be applied to the Kraichnan model to provide a validation of scaling complementary to the zero modes picture of Sec. 5.

11. INVERSE (INFRA-RED) RENORMALIZATION GROUP FOR THE KRAICHNAN MODEL

The idea to investigate the Kraichnan model by implementing a Wilson's infra-red recursion scheme was put forward first in Refs. 31, 32. There, it was argued that the Kraichnan model with a quasi-Lagrangian velocity field[(12)] had a inverse RG fixed point and scaling fields with dimensions given by the exponents found from the zero mode analysis of structure functions.

The purpose here it to carry out such an analysis in more detail to the canonical Eulerian representation of the Kraichnan model.

11.1. General Settings for Inverse Renormalization

It was observed in Refs. 31, 32 that one should consider the IRG transformation in the space of MSR actions without the forcing which should be treated as the correlation functions. The basic fields are

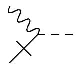
$$\phi = (\theta, \bar{\theta}, \mathbf{v}). \tag{130}$$

The starting point is the measure

$$e^{-\mathcal{A}(\phi)}\mathcal{D}[\phi] = e^{-\mathcal{A}_v(\theta,\bar{\theta},\mathbf{v})} d\mu_R(\theta, \bar{\theta}) d\mu_D(\mathbf{v}) \tag{131}$$

where

$$\mathcal{A}_v(\theta, \bar{\theta}, \mathbf{v}) := -\imath\langle \bar{\theta}, v^\alpha \partial_\alpha \theta \rangle = \tag{132}$$



and μ_D the Gaussian measure with covariance D and μ_R is the Gaussian "measure" with "covariance" given by the free response function (57). To keep to the RG formalism discussed in Sec. 10 we introduce to R the same infrared cutoff m as in the velocity covariance D. This is only for notational convenience as can easily be checked below.

The IRG transformation is defined by decomposing the fields as follows. The scaling exponents of time and the velocity are tied together by Galilean invariance, i.e. they are chosen so as to preserve the material derivative

$$\nabla_t = \partial_t + v^\alpha \partial_\alpha \tag{133}$$

under scaling. This means that

$$\eta_v = -1 - \eta_t. \tag{134}$$

The velocity field is decomposed into a scaling and fluctuating part

$$\mathbf{v} = \mathbf{v}'_{(1/\lambda)} + \delta\mathbf{v}. \tag{135}$$

It is readily seen that $\mathbf{v}'$ is again Gaussian with covariance

$$\prec v'^\alpha(\mathbf{x}, t) v'^\beta(\mathbf{y}, s) \succ = \lambda^{-2\eta_v - \eta_t - \xi} \delta(t - s) D^{\alpha\beta}(\mathbf{x}, m) \tag{136}$$

whereas the covariance of the velocity fluctuation denoted by

$$\delta \underbrace{v^\alpha(\mathbf{x}, t)\delta v}^\beta(\mathbf{y}, s) := \delta(t - s)\delta D^{\alpha\beta}(\mathbf{x} - \mathbf{y}, m) \tag{137}$$

is given as

$$\delta D^{\alpha\beta}(\mathbf{x}, m) = D_0 \xi \int \frac{d^d p}{(2\pi)^d} \frac{e^{\imath \mathbf{p}\cdot\mathbf{x}}}{p^{d+\xi}} \Pi_{\alpha\beta}(\hat{\mathbf{p}}) \chi_{[m,\lambda m]}(p). \tag{138}$$

Similar decomposition of the fields θ and $\bar{\theta}$

$$\theta = \theta'_{(1/\lambda)} + \delta\theta \tag{139a}$$

$$\bar{\theta} = \bar{\theta}'_{(1/\lambda)} + \delta\bar{\theta} \tag{139b}$$

leads to

$$\prec \theta'(\mathbf{x}, t)\bar{\theta}'(\mathbf{y}, s) \succ = \lambda^{-\eta_\theta - \eta_{\bar{\theta}} + d} R(\mathbf{x} - \mathbf{y}, t - s, \lambda^{2+\eta_t}\varkappa_\star) \equiv R'(\mathbf{x} - \mathbf{x}, t - s). \tag{140}$$

Thus we will fix

$$\eta_\theta + \eta_{\bar{\theta}} = d. \tag{141}$$

The fluctuation covariance is given by

$$\begin{aligned} \delta R(\mathbf{x} - \mathbf{y}, t - s) &\equiv \underbrace{\delta\theta(\mathbf{x}, t)\delta\bar{\theta}}(\mathbf{y}, s) \\ &= \imath H_0(t - s) \int \frac{d^d p}{(2\pi)^d} e^{\imath \mathbf{p}\cdot(\mathbf{x}-\mathbf{y}) - \frac{\varkappa_\star}{2} p^2 (t-s)} \chi_{[m,\lambda m]}(p). \end{aligned} \tag{142}$$

The IRG transformation of the measure (131) is given by

$$e^{-\mathcal{R}_\lambda \mathcal{A}(\phi)} \mathcal{D}[\phi'] = e^{-\mathcal{A}'_v(\theta', \bar{\theta}', \mathbf{v}')} d\mu_{R'}(\theta', \bar{\theta}') d\mu_{D'}(\mathbf{v}') \tag{143}$$

where

$$\mathcal{A}'_v(\theta', \bar{\theta}', \mathbf{v}') = -\log \int e^{-\mathcal{A}_v(\theta, \bar{\theta}, \mathbf{v})} d\mu_{\delta R}(\delta\theta, \delta\bar{\theta}) d\mu_{\delta D}(\delta\mathbf{v}) \tag{144}$$

and $\theta, \bar{\theta}$ and $\mathbf{v}$ have been decomposed according to (139a), (139b) and (135).

11.2. Infra-red Renormalization Group Flow of the Action

Let us do first the δv integral in (144)-

$$e^{-\mathcal{A}'_v(\phi')} = \int e^{\imath\langle\bar{\theta}, v'^\alpha \partial_\alpha \theta\rangle - \frac{1}{2}\langle\bar{\theta}\partial_\alpha\theta, \delta D^{\alpha\beta}\bar{\theta}\partial_\beta\theta\rangle} d\mu_{\delta R}(\delta\theta, \delta\bar{\theta}) \tag{145}$$

where

$$\theta = \theta'_{(1/\lambda)} + \delta\theta. \tag{146}$$

The second exponent on the RHS equals at $\delta\theta = 0 = \delta\bar{\theta}$

$$\langle\bar{\theta}'_{(1/\lambda)}\partial_\alpha\theta'_{(1/\lambda)}, \delta D^{\alpha\beta}\bar{\theta}'_{(1/\lambda)}\partial_\beta\theta'_{(1/\delta)}\rangle = \lambda^{2-\xi+\eta_t}\langle\bar{\theta}'\partial_\alpha\theta', \delta D^{\alpha\beta}_{(\lambda)}\bar{\theta}'\partial_\beta\theta'\rangle. \tag{147}$$

The large scale velocity is dominated by the constant mode of the velocity field

$$\delta D^{\alpha\beta}_{(\lambda)} = (\lambda^\xi - 1)\varkappa_\star\delta^{\alpha\beta} - \delta d^{\alpha\beta}_\star(\mathbf{x}, m) + o\left(\frac{1}{\lambda}\right) \tag{148}$$

where

$$\delta d^{\alpha\beta}_\star(\mathbf{x}, m) = D_0\xi \int \frac{d^d p}{(2\pi)^d} \frac{1 - e^{\imath\mathbf{p}\cdot\mathbf{x}}}{p^{d+\xi}} \Pi_{\alpha\beta}(\hat{\mathbf{p}})\chi_{[0,m]}(p). \tag{149}$$

Due to the unstable constant mode of the velocity field the only scaling under which the IRG stabilizes as λ tends to infinity is the Gaussian scaling

$$\eta_t = -2. \tag{150}$$

From (136) we see that the covariance D' then tends to zero i.e. the velocity field disappears from the action. In that limit then

$$e^{-\mathcal{A}'_v(\phi')} = \lim_{\lambda\to\infty} \int e^{-\frac{\varkappa_\star}{2}\int dt (\int dx \bar{\theta}\partial_\alpha \theta)^2} d\mu_{\delta R}(\delta\theta, \delta\bar{\theta}). \tag{151}$$

But

$$\int dx \bar{\theta}\partial_\alpha\theta = \int dx \bar{\theta}'_{(1/\lambda)}\partial_\alpha \theta'_{(1/\lambda)} + \int dx \delta\bar{\theta}\partial_\alpha \delta\theta \tag{152}$$

due to the disjoint supports of $\bar{\theta}'_{1/\lambda}$ and $\delta\bar{\theta}$ in momentum space. Thus we end up with the fixed point (up to a constant)

$$\mathcal{A}^\star_v = \frac{\varkappa_\star}{2} \int dt \left(\int dx \bar{\theta}\partial_\alpha \theta \right)^2. \tag{153}$$

The action (153) coincides with

$$\mathcal{A}^\star_v = -\ln \prec e^{\imath \langle \bar{\theta}, v^\alpha_\star \partial_\alpha \theta \rangle} \succ_{v^\star} \tag{154}$$

where $\mathbf{v}_\star$ is a velocity field with constant covariance

$$\prec v^\alpha_\star(t) v^\beta_\star(s) \succ = \delta(t-s)\varkappa_\star \delta^{\alpha\beta} \tag{155}$$

with $\varkappa_\star$ given by (17). The only nonzero correlations of the fixed point theory are the multiple response functions

$$\prec \prod_{i=1}^n \theta(\mathbf{x}_i, t_i)\bar{\theta}(\mathbf{y}_i, s_i) \succ = \prod_{i=1}^n \frac{\delta}{\delta \jmath(\mathbf{x}_i, t_i)} \frac{\delta}{\delta \bar{\jmath}(\mathbf{y}_i, s_i)} \bigg|_{\jmath=\bar{\jmath}=0} \prec e^{\imath \langle \jmath, \Re H \bar{\jmath} \rangle} \succ_{\mathbf{v}} \tag{156}$$

and thus they coincide with the velocity averages of the same response functions in a random constant velocity field.

Using Hopf equations (27) the response functions of the Kraichnan model with only two times are given as heat kernels

$$\prec \prod_{i=1}^n \theta(\mathbf{x}_i, t)\bar{\theta}(\mathbf{y}_i, s) \succ = H_0(t-s) e^{\mathcal{M}_n(t-s)}(\mathbf{x}_1, \ldots, \mathbf{x_n}, \mathbf{y}_1, \ldots, \mathbf{y}_n) \tag{157}$$

of the operators

$$\mathcal{M}_n = \frac{\kappa}{2} \sum_i \partial^2_{\mathbf{x}_i} + \frac{D m^{-\xi}}{2} \left(\sum_i \partial_{\mathbf{x}_i} \right)^2 + \sum_{i<j} d^{\alpha\beta} \partial_{\mathbf{x}^\alpha_i} \partial_{\mathbf{x}^\beta_j} \tag{158}$$

and the multiple time ones are simple combinations of these with various n's. Under the scaling of space and time where $\eta_t = -2$ the $d^{\alpha\beta}$ will contract by $L^{-\xi}$ thus explaining the fixed point.

The reason for the trivial fixed point of the IRG lies, as stressed in Refs. 31, 32, in the constant mode of the velocity field. As we saw in Sec. 4 this mode decouples in the stationary equal time correlators i.e. the second term drops out from the operators (158) when acting on translationally invariant functions. However, it doesn't decouple in the unequal time stationary correlators. For example, the two-point function of the scalar is given by

$$\prec \theta(\mathbf{x}, t)\theta(\mathbf{y}, s) \succ \; = \; \prec \left[e^{\mathcal{M}_1(t-s)}\theta(\mathbf{x}, s)\right] \theta(\mathbf{y}, s) \succ \tag{159}$$

and the time dependence is dominated by the eddy diffusivity $\varkappa_\star = Dm^{-\xi}$.

It is possible to modify the velocity covariance so that the zero mode decouples also from the time evolution while the stationary state is left unchanged,[31,32] an example being the quasi-Lagrangean velocity field. Then one may choose scaling $\eta_t = -2 + \xi$ and the IRG fixed point is less trivial. However, in both cases the scaling properties of the fixed point action bear very little information about the structure functions $\mathcal{S}_{2n}$. To understand their scaling we need to find the relevant scaling fields. In the next section we show how they appear in our model of Eulerian velocities thus bypassing the more involved quasi-Lagrangean formalism proposed in Refs. 31, 32.

12. INFRA-RED RENORMALIZATION GROUP ANALYSIS OF THE STRUCTURE FUNCTIONS

By definition structure functions are the averages

$$\mathcal{S}_{2n}(\mathbf{x}) \; = \prec [\theta(\mathbf{x}) - \theta(0)]^{2n} \frac{[\imath\langle\bar{\theta}, f\rangle]^{2n}}{(2n)!} \succ \tag{160}$$

with respect to the measure (131) and the Gaussian measure of the forcing field f. For the renormalization group calculations it is convenient to take the forcing covariance (2) as

$$\prec \check{f}(\mathbf{p}_1, t_1)\check{f}(\mathbf{p}_2, t_2) \succ \; = \; F_\star \frac{\delta(p_1 - m_F)}{m_F^{d-1}} \delta^{(d)}(\mathbf{p}_1 + \mathbf{p}_2)\delta(t - s) \tag{161}$$

where $m_F = L_F^{-1}$. The forcing is in such a case isotropic. An example of anisotropic forcing is

$$\prec \check{f}(\mathbf{p}_1, t_1)\check{f}(\mathbf{p}_2, t_2) \succ = F_\star \frac{\delta^{(d)}(\mathbf{p}_1 - \mathbf{q}^\star) + \delta^{(d)}(\mathbf{p}_1 + \mathbf{q}^\star)}{2} \delta^{(d)}(\mathbf{p}_1 + \mathbf{p}_2)\delta(t - s) \tag{162}$$

with $|q|^\star = m_F$.

As in Secs. 8 and 9 we work under the assumption

$$\frac{m_F}{m} \ll 1. \tag{163}$$

It should be stressed that this is for simplicity only and the conclusions of the renormalization group analysis of the structure function hold true for an arbitrary value of the ratio of the integral scales provided the requirement of infra-red forcing is satisfied.(45) In the language of renormalization group this latter requirement means that

$$f = \delta f \tag{164}$$

which obviously is true as soon as $m_F \in [0, \lambda m]$. Thus the structure functions are given as the expectation value

$$S_{2n}(\mathbf{x}; \phi) = [\mathcal{J}(\mathbf{x})]^{2n} \frac{\imath^{2n} \langle \delta\bar{\theta}, F\delta\bar{\theta}\rangle^n}{2^n n!} \tag{165}$$

where F is the spatial part of the forcing correlation and using invariance under $\mathbf{x} \to -\mathbf{x}$ we replaced the scalar increment by

$$\mathcal{J}(\mathbf{x}) = \frac{1}{2}(\theta(\mathbf{x}, 0) + \theta(-\mathbf{x}, 0) - 2\theta(0, 0)) \tag{166}$$

In the limit m_F tending to zero, the structure function operator involves of powers of

$$\lim_{m_F \downarrow 0} \langle \delta\bar{\theta}, F\delta\bar{\theta}\rangle \propto \int_{-\infty}^{\infty} dt \delta\check{\bar{\theta}}(0, t) \delta\check{\bar{\theta}}(0, t) \tag{167}$$

which is a field functional *local in momentum space.*

12.1. Scaling Field for the Structure Functions

At zeroth order in ξ the isotropic component of the renormalized structure function has the scaling limit

$$\lim_{\lambda \uparrow \infty} \lambda^{\zeta_{2n,j}} \int d\Omega_d Y^*_{j0}(\hat{\mathbf{x}})(\mathcal{L}_\lambda S_{2n}) \left(\frac{\mathbf{x}}{\lambda}; \phi\right) = \int d\Omega_d Y^*_{j0}(\hat{\mathbf{x}}) \mathcal{S}^{(0)}_{2n}(\mathbf{x}) \tag{168}$$

for

$$\zeta_{2n,j} = 2n. \tag{169}$$

12.1.1. *First Order in* ξ

We will work out perturbative corrections to (168) in the limit $m_F \to 0$. In this limit there is no momentum flow along scalar correlation lines even at finite

λ. In consequence, couplings generated by Wick contractions of scalar fields in the interaction vertex (132) with ghost fields in the forcing vertex (167) factorize as

$$\begin{aligned}&\langle \bar{\theta}_{(1/\lambda)}, v^{\alpha}_{(1/\lambda)} \partial_{\alpha} \delta\theta \rangle \langle \delta\theta, F \delta\theta \rangle \delta\mathcal{J}\left(\frac{\mathbf{x}}{\lambda}\right) \\ &\quad = \langle \bar{\theta}_{(1/\lambda)}, v^{\alpha}_{(1/\lambda)} \rangle \int \frac{d^d p}{(2\pi)^d} i p_{\alpha} \frac{2\cos\left(\frac{\mathbf{x}}{\lambda}\cdot \mathbf{p}\right) - 1}{2p^2} \check{F}(p) \end{aligned} \tag{170}$$

which vanishes because of parity (recall that $\check{F}(p) \propto \delta(p)$ as $m_F \to 0$). At leading order in ξ the scaling field associated to the structure function operation is given by

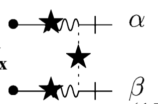
$$\begin{aligned}&\lim_{\lambda\uparrow\infty} \lambda^{\zeta_{2n,j}} \int d\Omega_d Y^*_{j0}(\hat{\mathbf{x}}) (\mathcal{L}_{\lambda} S_{2n})\left(\frac{\mathbf{x}}{\lambda};\phi\right) \\ &\quad = \lim_{\lambda\uparrow\infty} \int d\Omega_d Y^*_{j0}(\hat{\mathbf{x}}) \left\{ (1 + \zeta^{(1)}_{2n,0} \ln\lambda) \mathcal{S}_{2n}(\mathbf{x}) + \xi \lambda^{\zeta^{(0)}_{2n}} \frac{d}{d\xi}\bigg|_{\xi=0} (\mathcal{L}_{\lambda} S_{2n})\left(\frac{\mathbf{x}}{\lambda};\phi\right) \right\} + O(\xi^2)\end{aligned} \tag{171}$$

where

$$\begin{aligned}&\frac{d}{d\xi}\bigg|_{\xi=0} (\mathcal{L}_{\lambda} S_{2n})(\mathbf{x};\phi) \\ &\quad = \frac{1}{2}\binom{2n}{2n-2} \mathcal{S}_{2n-2}(\mathbf{x}) \left[\delta\mathcal{V}^{(0)\alpha\beta}_{(1;4)}(\mathbf{x},\lambda) + \mathfrak{V}^{(0)\alpha\beta}_{(1;4)}(\mathbf{x},\phi_{(1/\lambda)}) \right] (\partial_{\alpha}\partial_{\beta}\mathcal{S}_2)(\mathbf{x}) \\ &\qquad + 3\binom{2n}{2n-4} \mathcal{S}_{2n-4}(\mathbf{x}) \left[\delta\mathcal{V}^{(0)\alpha\beta}_{(1;4)}(\mathbf{x},\lambda) + \mathfrak{V}^{(0)\alpha\beta}_{(1;4)}(\mathbf{x},\phi_{(1/\lambda)}) \right] (\partial_{\alpha}\mathcal{S}_2)(\mathbf{x}) (\partial_{\beta}\mathcal{S}_2)(\mathbf{x})\end{aligned} \tag{172}$$

The vertices are given by the small scale field functional

$$\mathfrak{V}^{\alpha\beta}_{(1;4)}\left(\mathbf{x},\phi_{(1/\lambda)}\right) = \mathcal{J}^2_{(1/\lambda)}(\mathbf{x}) \langle \bar{\theta}_{(1/\lambda)}, v^{\alpha}_{(1/\lambda)} \rangle \langle \bar{\theta}_{(1/\lambda)}, v^{\beta}_{(1/\lambda)} \rangle := \mathcal{I}_{\mathbf{x}}$$

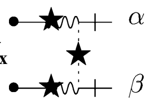


(173)

and the averaged value of its large scale (small momentum) counterpart

$$\delta\mathcal{V}^{\alpha\beta}_{(1;4)}(\mathbf{x},\lambda) = \frac{2D_0 m^{\xi}}{D} \int \frac{d^d p}{(2\pi)^d} \frac{1 - e^{\imath \mathbf{p}\cdot\mathbf{x}}}{p^2} \frac{\Pi^{\alpha\beta}(\hat{p})}{p^{d+\xi}} \chi_{[m,\lambda m]}(p)$$

$$= \mathfrak{P}_{\lambda} \mathcal{I}_{\mathbf{x}}$$



(174)

whilst the notation (77) is adopted for derivatives with respect to the Hölder exponent ξ. The symbol $\mathfrak{P}_{\lambda}$ in (174) denotes the restriction of the momentum support of the velocity correlation. As λ tends to infinity the diagram has the limit

$$\lambda^2 \delta\mathcal{V}^{(0)\alpha\beta}_{(1;4)}\left(\frac{\mathbf{x}}{\lambda}, \lambda\right) \overset{\lambda\uparrow\infty}{=} \ln\lambda \frac{(d+1)x^2}{(d-1)(d+2)} \mathcal{T}^{\alpha\beta}(\hat{\mathbf{x}}, 2) + \delta\mathcal{V}^{(0)\alpha\beta}_{\star(1;4)}(\mathbf{x}) \tag{175}$$

Little algebra yields the final form of (171)

$$\lim_{\lambda\uparrow\infty} \lambda^{\zeta_{2n,j}} \int d\Omega_d Y^*_{j0}(\hat{\mathbf{x}})(\mathcal{L}_\lambda S_{2n})(\mathbf{x};\phi) = \int d\Omega_d Y^*_{j0}(\hat{\mathbf{x}})\mathcal{S}_{2n}(\mathbf{x})$$
$$+ \frac{\xi}{2} \int d\Omega_d Y^*_{j0}(\hat{\mathbf{x}}) \left[\delta\mathcal{V}^{(0)\alpha\beta}_{\star(1;4)}(\mathbf{x}) + \mathfrak{V}^{(0)\alpha\beta}_{(1;4)}(\phi)\right] \partial_\alpha\partial_\beta \mathcal{S}_{2n}(\mathbf{x}) + O(\xi^2) \tag{176}$$

provided $\zeta_{2n,j}$ is given by (84).

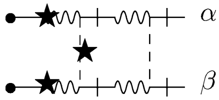
12.1.2. Second Order in ξ

The calculation of the structure function scaling field can be inferred from the perturbation theory of Sec. 8. The field independent part of the scaling field is given by the structure function diagrams in the presence of a fixed arbitrary ultra-violet cutoff. The field dependent part is obtained by pruning lines in these diagrams and replacing them with the corresponding pair of ultra-violet scaling fields. Accordingly, up to second order accuracy in ξ, the renormalized structure function takes the form

$$(\mathcal{L}_\lambda S_{2n})(\mathbf{x};\phi)$$
$$= \left\{1 + \frac{\xi}{2}\sum_{r=0}^{1} \xi^r \left[\mathfrak{V}^{(r)\alpha\beta}_{(1;4)}(\mathbf{x}, \phi_{(1/\lambda)}) + \delta\mathcal{V}^{(r)\alpha\beta}_{(1;4)}(\mathbf{x},\lambda)\right]\right\} \partial_\alpha\partial_\beta \mathcal{S}^{(0)}_{2n}(\mathbf{x})$$
$$+ \frac{\xi^2}{2}\left\{\mathfrak{V}^{(0)\alpha\beta}_{(2;4)}(\mathbf{x}, \phi_{(1/\lambda)}) + \delta\mathcal{V}^{(0)\alpha\beta}_{(2;4)}(\mathbf{x},\lambda)\right\} \partial_\alpha\partial_\beta \mathcal{S}^{(0)}_{2n}(\mathbf{x})$$
$$+ \xi^2\left\{\mathfrak{V}^{(0)\alpha\beta\mu}_{(2;6)}(\mathbf{x}, \phi_{(1/\lambda)}) + \delta\mathcal{V}^{(0)\alpha\beta\mu}_{(2;6)}(\mathbf{x},\lambda)\right\} \partial_\alpha\partial_\beta\partial_\mu \mathcal{S}^{(0)}_{2n}(\mathbf{x})$$
$$+ \frac{\xi^2}{8}\left\{\mathfrak{V}^{(0)\alpha\beta\mu\nu}_{(2;8)}(\mathbf{x}, \phi_{(1/\lambda)}) + \delta\mathcal{V}^{(0)\alpha\beta\mu\nu}_{(2;8)}(\mathbf{x},\lambda)\right\} \partial_\alpha\partial_\beta\partial_\mu\partial_\nu \mathcal{S}^{(0)}_{2n}(\mathbf{x}) + O(\xi^3, L_F^{-2}). \tag{177}$$

The $\delta\mathcal{V}^{\bullet}_{\bullet}$'s denote the diagrams (87), (88), and (88) where velocity correlations have momentum support in $[m, \lambda m]$. Infra-red logarithmic behavior of the field dependent vertices is identified by considering the scaling limits

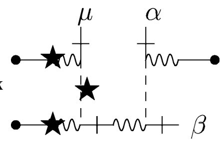
$$\lim_{\lambda\uparrow\infty} \lambda^2 \mathfrak{V}^{\alpha\beta}_{(2;4)}\left(\frac{\mathbf{x}}{\lambda}, \phi_{(1/\lambda)}\right) = \lim_{\lambda\uparrow\infty} \lambda^2 \mathcal{I}_{\mathbf{x}}$$



$$= \frac{2(d+1)\ln\lambda}{(d-1)(d+2)} \mathfrak{V}^{\mu\nu}_{(1;4)}(\mathbf{x},\phi)\mathcal{Q}^{\alpha\beta}_{\mu\nu} + O(\lambda^0,\xi) \tag{178}$$

and

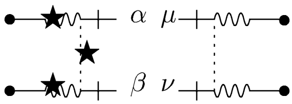
$$\lim_{\lambda\uparrow\infty} \lambda^3 \mathfrak{V}^{\alpha\beta;\mu}_{(2;6)}\left(\frac{\mathbf{x}}{\lambda}, \phi_{(1/\lambda)}\right) = \lim_{\lambda\uparrow\infty} \lambda^3 \mathcal{I}_{\mathbf{x}}$$



$$= \frac{2(d+1)\ln\lambda}{(d-1)(d+2)} \mathfrak{V}^{\mu\rho}_{(1;4)}(\mathbf{x},\phi)\mathcal{Q}^{\alpha\beta}_{\rho\sigma}x^\sigma + O(\lambda^0,\xi) \tag{179}$$

and

$$\lim_{\lambda\uparrow\infty} \lambda^4 \mathfrak{V}^{\alpha\beta;\mu\nu}_{(2;8)}\left(\frac{\mathbf{x}}{\lambda}, \phi_{(1/\lambda)}\right) = \lim_{\lambda\uparrow\infty} \lambda^4 \mathcal{I}_{\mathbf{x}}$$



$$= \frac{4(d+1)\ln\lambda}{(d-1)(d+2)} \mathfrak{V}^{\alpha\beta}_{(1;4)}(\mathbf{x},\phi)\mathcal{Q}^{\mu\nu}_{\rho\sigma}x^\rho x^\sigma + O(\lambda^0,\xi). \tag{180}$$

The tensor structure of the diagrams is specified by

$$\mathcal{Q}^{\alpha\beta}_{\mu\nu} := \frac{1}{2}\partial_\mu\partial_\nu x^2 \mathcal{T}^{\alpha\beta}(\mathbf{x},2) = \delta^{\alpha\beta}\delta_{\mu\nu} - \frac{\delta^\alpha_\mu\delta^\beta_\nu + \delta^\alpha_\nu\delta^\beta_\mu}{d+1}. \tag{181}$$

Only the index symmetric part of the above diagrams contributes to (177) owing to the contraction with fully symmetric quantities. As in Secs. (8) and Secs. (9) this fact is emphasized by omitting semicolons between non-symmetric indices.

In Sec. (8) it was shown that up to second order in ξ the perturbative expression of any structure function in the inertial range is compatible with that of a homogeneous function. This information together with the scaling limits (178), (179) and (180) permit to verify after some straightforward algebra that the structure function operator has a finite scaling limit

$$\lim_{\lambda\uparrow\infty} \lambda^{\zeta_{2n,j}} \int d\Omega_d Y^*_{j0}(\hat{\mathbf{x}})(\mathcal{L}_\lambda S_{2n})\left(\frac{\mathbf{x}}{\lambda};\phi\right) = \left(\sum_{r=0}^{2}\frac{\xi^r}{r!}\frac{d^r\lambda^{\zeta_{2n,j}}}{d\xi^r}\right)\int d\Omega_d Y^*_{j0}(\hat{\mathbf{x}})\mathcal{S}^{(0)}_{2n}(\mathbf{x})$$
$$+\,\xi\left(\sum_{r=0}^{1}\frac{\xi^r}{r!}\frac{d^r\lambda^{\zeta_{2n,j}}}{d\xi^r}\right)\int d\Omega_d Y^*_{j0}(\hat{\mathbf{x}})\left\{S^{(1)}_{2n}(\mathbf{x};\phi) - \mathcal{S}^{(0)}_{2n}(\mathbf{x})\zeta^{(1)}_{2n}\ln\lambda\right\}$$

$$+ \frac{\xi^2}{2} \int d\Omega_d Y^*_{j0}(\hat{\mathbf{x}}) \Big\{ S^{(2)}_{2n}(\mathbf{x};\phi) - 2S^{(1)}_{2n}(\mathbf{x};\phi)\zeta^{(1)}_{2n;j} \ln\lambda$$

$$+ \mathcal{S}^{(0)}_{2n}(\mathbf{x}) \Big[\big(\zeta^{(1)}_{2n;j} \ln\lambda\big)^2 - \zeta^{(2)}_{2n;j} \ln\lambda \Big] \Big\}$$

$$+ O(\xi^3) = S^{\star}_{2n;j}(\mathbf{x};\phi) \tag{182}$$

with $S^{(i)}_{2n}(\mathbf{x};\phi)$, $i = 1, 2$ obtained by gathering the field dependence in (177) according to its asymptotic behavior in λ and provided the scaling dimension $\zeta_{2n,j}$ is specified by (93).

The conclusion of the above analysis is that the infra-red renormalization of the structure function operator produces a well defined scaling field. It is worth noting that this conclusion holds also had we averaged out the velocity field from the outset. The resulting scaling field will then depend only on the ultra-violet degrees of freedom of the scalar and ghost fields but it has the same scaling exponent.

13. ULTRA-VIOLET RENORMALIZATION GROUP

Ultra-violet renormalization group addresses the question of removing the ultraviolet cutoff in the theory stemming from the one (M) in the velocity covariance. Although the correlation functions of the θ fields have a well defined $M \to \infty$ limit the gradients will not have as was discussed in Sec. 9. Ultra-violet renormalization will study that divergence by finding the appropriate scaling fields.

Ultra-violet renormalization was applied to the Kraichnan model in Refs. 2, 3, 5 in the framework of the minimal subtraction scheme.[19,40,51,59] The minimal subtraction scheme has the merit to provide probably the most computationally efficient setting for the determination of the scaling exponents $\zeta_{2n,j}$ which in Refs. 2, 3 were determined up to third order in ξ.

The purpose of the present section is to reproduce to leading order in ξ the same calculation using Wilson's original scheme in order to render the comparison with infra-red renormalization more transparent.

13.1. The Effective Action

In order to inquire the limit M tending to infinity, it is more convenient not to use rescalings in the renormalization group, i.e. to proceed as in Eq. (114). The fluctuation covariances have momenta on $[\bar{m}, M]$ and they become in the limit of infinite ultra-violet cutoff

$$\lim_{M\uparrow\infty} \delta R(\mathbf{x}, t) = \imath H_0(t) \int \frac{d^d p}{(2\pi)^d} e^{\imath \mathbf{p}\cdot\mathbf{x} - \frac{\varkappa_\star}{2} p^2 t} \chi_{[\bar{m},\infty]}(p) \tag{183}$$

and

$$\lim_{M\uparrow\infty} \delta D^{\alpha\beta}(\mathbf{x}) = \xi D_0(t) \int \frac{d^d p}{(2\pi)^d} \frac{e^{\imath p\cdot\mathbf{x}}}{p^{d+\xi}} \Pi^{\alpha\beta}(p) \chi_{[\bar{m},\infty]}(p). \tag{184}$$

Since the main interest is to determine the statistical properties of the scalar and the ghost fields, the renormalization group transformation will be applied directly to the interaction term obtained by averaging out all the degrees of freedom of the velocity field. The effective action at scale $\bar{m}$ stabilizes trivially as $M \to \infty$ at all orders in perturbation theory. At first order it is simply given by

$$\mathcal{A}_{\bar{m}} = \frac{\xi}{2} \langle \bar{\theta} \partial_\alpha \theta, D^{(0)\alpha\beta} \bar{\theta} \partial_\beta \theta \rangle + o(\xi^2) \tag{185}$$

with

$$D^{(r)\alpha\beta} = \frac{d^r}{d\xi^r}\bigg|_{\xi=0} D^{\alpha\beta}. \tag{186}$$

13.2. Ultra-Violet Renormalization Group Analysis of Radial Gradients

More interesting flow is found once we look at the scaling fields involving gradients of the scalar. Let $\mathcal{L}_{\bar{m}}$ be the linearization of the above renormalization group in the limit as M tends to infinity. Acting on radial gradients ultraviolet renormalization gives

$$\begin{aligned} \mathcal{L}_{\bar{m}}[\hat{\mathbf{x}} \cdot \partial\theta(\mathbf{y}, t)]^{2n} = {} & [\hat{\mathbf{x}} \cdot \partial\theta(\mathbf{y}, t)]^{2n} \\ & -\imath\xi \binom{2n}{1} [\hat{\mathbf{x}} \cdot \partial\theta(\mathbf{y}, t)]^{2n-1} [\hat{\mathbf{x}} \cdot \partial\delta\underline{\theta(\mathbf{y}, t)] \langle \delta\bar{\theta}} \partial_\alpha\theta, D^{(0)\alpha\beta} \bar{\theta} \partial_\beta \theta \rangle \\ & -\xi \binom{2n}{2} [\hat{\mathbf{x}} \cdot \partial\theta(\mathbf{y}, t)]^{2n-2} [\hat{\mathbf{x}} \cdot \partial\delta\theta(\mathbf{y}, t)]^2 \langle \delta\bar{\theta} \partial_\alpha\theta, D^{(0)\alpha\beta} \delta\bar{\theta} \partial_\beta \theta \rangle \\ & + O(\xi^2). \end{aligned} \tag{187}$$

In order to streamline the notation, on the right hand side of (187) coarse grained fields with ultra-violet cutoff $\bar{m}$ are represented by the letters $\theta, \bar{\theta}$. Leading order corrections are specified by the couplings

$$\begin{aligned} & [\hat{\mathbf{x}} \cdot \partial\delta\underline{\theta(\mathbf{y}, t)] \langle \delta\bar{\theta}} \partial_\alpha\theta, D^{(0)\alpha\beta} \bar{\theta} \partial_\beta \theta \rangle \\ & = \int_{-\infty}^{t} ds \prod_{i=1}^{2} d^d y_i [\hat{\mathbf{x}} \cdot (\partial_{\mathbf{y}} \delta R)(\mathbf{y} - \mathbf{y}_1, t - s)] \\ & \times D^{\alpha_1\alpha_2}(\mathbf{y}_1 - \mathbf{y}_2) \bar{\theta}(\mathbf{y}_j, s) \prod_{j=1}^{2} \partial_{\alpha_j} \theta(\mathbf{y}_j, s) \end{aligned} \tag{188}$$

and

$$[\hat{\mathbf{x}} \cdot \partial \delta\theta(\mathbf{y}, t)]^2 \langle \delta\bar{\theta}\partial_\alpha\theta, D^{(0)\alpha\beta}\delta\bar{\theta}\partial_\beta\theta\rangle$$
$$= \int_{-\infty}^{t} ds \prod_{i=1}^{2} d^d y_i D^{(0)\alpha_1\alpha_2}(\mathbf{y}_1 - \mathbf{y}_2) \prod_{j=1}^{2} [\hat{\mathbf{x}} \cdot (\partial_{\mathbf{y}}\delta R)(\mathbf{y} - \mathbf{y}_j, t - s)]\partial_{\alpha_j}\theta(\mathbf{y}_j, s). \tag{189}$$

Let us shift the variables $\mathbf{y}_i \to \mathbf{y}_i + \mathbf{y}$ and Taylor expand

$$\theta(\mathbf{y} + \mathbf{y}_i, t + s) = \sum_{n,k=0}^{\infty} \frac{1}{n!k!}[(\mathbf{y}_i \cdot \partial_{\mathbf{y}})^n (s\partial_t)^k \theta](\mathbf{y}, t) \tag{190}$$

for $i = 1, 2$. Then the leading order of this expansion gives the most singular part of the integral as the ultra-violet cut-off tends to infinity. Higher orders improve the ultra-violet behavior of the integral. Thus it is found

$$[\hat{\mathbf{x}} \cdot \partial \delta\theta(\mathbf{y}, t)]\langle \delta\bar{\theta}\partial_\alpha\theta, D^{(0)\alpha\beta}\bar{\theta}\partial_\beta\theta\rangle$$
$$= \int_{0}^{\infty} ds \prod_{i=1}^{2} d^d y_i \hat{\mathbf{x}} \cdot (\partial_y \delta R)(\mathbf{y}_1, s) D^{\alpha_1\alpha_2}(\mathbf{y}_1 - \mathbf{y_2})$$
$$+ O\left(\frac{1}{M^2}, \frac{1}{\bar{m}^2}\right) = O\left(\frac{1}{M^2}, \frac{1}{\bar{m}^2}\right) \tag{191}$$

by parity and

$$[\hat{\mathbf{x}} \cdot \partial \delta\theta(\mathbf{y}, t)]^2 \langle \delta\bar{\theta}\partial_\alpha\theta \ , D^{(0)\alpha\beta}\delta\bar{\theta}\partial_\beta\theta\rangle$$
$$= \mathcal{U}^{(0)\alpha\beta}_{(1:4)}(\bar{m}, M)[\partial_\alpha\theta(\mathbf{y}, t)][\partial_\beta\theta(\mathbf{y}, t)] + O\left(\frac{1}{M^2}, \frac{1}{\bar{m}^2}\right) \tag{192}$$

where $\mathcal{U}^{\alpha\beta}_{(1:4)}$ was given in (104) and it is here evaluated in the momentum range $[\bar{m}, M]$. The terms neglected are irrelevant for the determination of the scaling dimension. This latter is exhibited in the scaling limit which in the present context can be taken by fixing the ratio

$$0 < \lambda = \frac{\bar{m}}{M} < 1 \tag{193}$$

and by considering

$$\lim_{M\uparrow\infty} \lambda^{\eta_{\mathcal{G}_{2n,j}}} \int d\Omega Y_{j0}(\hat{\mathbf{x}})\mathcal{L}_{\lambda M}[\hat{x} \cdot \partial\theta]^{2n} =$$
$$A_n ||\partial\theta||^{2n} \left\{1 + \xi \left[\eta^{(1)}_{\mathcal{G}_{2n,j}} - \frac{n(d+2n)}{d+2} + \frac{(d+1)j(d+j-2)}{2(d-1)(d+2)}\right] \ln\lambda\right\} + O(\xi^2) \tag{194}$$

where

$$\eta^{(1)}_{\mathcal{G}_{2n,j}} := \left.\frac{d}{d\xi}\right|_{\xi=0} \eta_{(\partial\theta)^{2n}}. \tag{195}$$

The derivation of (194) exploits the integral identity

$$\int d\Omega Y_{j,0}(\hat{\mathbf{x}})[\hat{\mathbf{x}} \cdot \partial\theta(\mathbf{y},t)]^{2n} = A_{n,j}[\partial\theta \cdot \partial\theta]^n(\mathbf{y},t) \tag{196}$$

with A_n satisfying (see appendix E for the proof)

$$(d+1)\frac{A_{n-1,j}}{A_{n,j}} - 2 = \frac{d-1}{2n-1}\left[d + 2n - \frac{(d+1)j(d+j-2)}{2n(d-1)}\right]. \tag{197}$$

Existence of scaling requires the cancellation of the λ dependence in (194) whence it follows

$$\eta^{(1)}_{\mathcal{G}_{2n,j}} = \frac{n(d+2n)}{d+2} - \frac{(d+1)j(d+j-2)}{2(d-1)(d+2)}. \tag{198}$$

Thus, $\|\partial\theta\|^{2n} + O(\xi^2)$ is a scaling field and the isotropic component of the radial gradients scales with the ultraviolet cut-off

$$\int d\Omega Y_{j0}(\hat{\mathbf{x}}) \prec (\hat{\mathbf{x}} \cdot \partial\theta)^{2n} \succ \sim M^{n\xi}\left(\frac{M}{m}\right)^{\rho_{2n,j}} \tag{199}$$

with $\rho_{2n,j}$ given (85).

Let us finish this section by comparing our treatment of the UV problem with that of Refs. 2, 3, 5. In these papers the anomalous exponents of the scalar gradients are computed using the field theoretic RG derived from dimensionally regularized perturbation expansion. Their starting point for the MSR theory differs from ours in two ways. First their velocity covariance (4) has $D_0\xi$ replaced by a constant with no explicit ξ dependence. Second, perturbation theory is done to the Stratonovich representation of the model (1).

The first difference means that the "bare" correlation functions $\mathcal{C}_{2n}$ need to be multiplied by a "renormalization constant" proportional to ξ^{-2n} to get a nontrivial limit as $\xi \to 0$.

The second difference leads to a logarithmic UV divergence at $\xi = 0$ which can be traced to the M dependence of the effective diffusivity κ in (26). Since the authors work in dimensional regularization an UV cutoff doesn't enter but its role is played by ξ that plays the same role as $d - 4$ in dimensional regularization. Using the Ito representation in the perturbation as we do the logarithmically UV divergent tadpole diagram doesn't enter and the only divergences to be dealt with are infrared. In both approaches the UV problem for the action is trivial, indeed, the authors find a simple fixed point ξ for a running coupling constant describing the strength of the nonlinearity and their perturbation expansion becomes an expansion in powers of ξ.

14. CONCLUSIONS

The Kraichnan model has the rare feature to allow for detailed analytical study of a turbulent system. In particular it allows to address the question of what kind of renormalization group if any is appropriate for turbulence, the traditional ultraviolet one or the more exotic inverse or infrared one.

In the context of the Kraichnan model both direct and inverse renormalization can be successfully applied. Direct renormalization is natural when studying the short distance singularities that appear when the dissipative scale is taken to zero. The scaling fields are local operators in the derivatives of the scalar and the exponents may be computed using various versions of the UV RG, in dimensional regularization as in Refs. 2, 3, 5, 55 or in the Wilsonian framework as in the present paper.

The inverse RG appears more natural for dealing with inertial range quantities such as the nonlocal operators that enter the study of the structure functions. The scaling fields are now very nonlocal in position space. We implemented the inverse RG in the Wilsonian framework. Although cumbersome it has the advantage of being conceptually clear. However other more computationally effective inverse schemes should be possible too.

A natural question is whether infra-red renormalization may be of use in the analysis of physical models other than Kraichnan's. It is important here to stress that the possibility of successfully applying infra-red renormalization ultimately relies on the physics of the system. It remains a challenge for the future to establish whether such tool may prove useful to inquire scaling properties of systems for which direct renormalization cannot be applied.

APPENDICES

A. MELLIN TRANSFORM

We use the following definition of the Mellin transform of a function $f : [0, \infty] \to \mathbf{R}$:

$$\tilde{f}(x, z) = \int_0^\infty \frac{dw}{w} \frac{f(wx)}{w^z} = x^z \int_0^\infty \frac{dw}{w} \frac{f(w)}{w^z} := x^z J(z). \tag{A.1}$$

Suppose f is given by

$$f(x) = x^a g(x) \tag{A.2}$$

with g decaying faster than any power at infinity. Then the Mellin transform is defined and analytic for $\Re z < a$. It has a pole at a. Indeed, write

$$\tilde{f}(x;z) = x^z \int_0^\infty \frac{dw}{w}\frac{g(w)}{w^{z-a}} = -\frac{x^z}{z-a}\int_{-\infty}^\infty du\, e^{-u} g(e^{\frac{u}{z-a}}) \tag{A.3}$$

so the residue is given by

$$Res\,\tilde{f}(x;a) = -x^z g(0). \tag{A.4}$$

If g is smooth at origin further "*ultra-violet*" poles will occur in $a+n$ for any integer n, in correspondence with the Taylor expansion of g. If g has only power law decay at infinity, say

$$g(x) \sim x^{-b}, \qquad b > 0 \tag{A.5}$$

the Mellin transform will exhibit "*infra-red*" poles also in $-b-n, n = 0, \ldots$.

The inverse Mellin transform is

$$f(x) = \int_{c-i\infty}^{c+i\infty} \frac{dz}{(2\pi\imath)} x^z J(z) \tag{A.6}$$

with c less than a. The inverse is usually computed by Cauchy's residue theorem. Finally for a function $f(\mathbf{x})$ on $\mathbf{R}^n$ we define

$$\tilde{f}(\mathbf{x}, z) = \int_0^\infty \frac{dw}{w}\frac{f(w\mathbf{x})}{w^z} x^2 \int_0^\infty \frac{dw}{w}\frac{f(w\hat{X})}{w^2} \tag{A.7}$$

with $x = |\mathbf{x}|$.

More details can be found for example in the textbook.[(49)] Useful examples of application of the Mellin transform to diagrammatic expansions in field theory are given also in Ref. 48.

B. EVALUATION OF THE INERTIAL RANGE ASYMPTOTICS OF THE VELOCITY FIELD

An algorithmically efficient evaluation of (18) is achieved using the integral representation of a power law:

$$\frac{1}{x^z} = \int_0^\infty \frac{du}{u}\frac{u^{\frac{z}{2}} e^{-ux^2}}{\Gamma\left(\frac{z}{2}\right)}, \qquad z > 0. \tag{B.1}$$

Namely, for negative values of z the order of integration in (18) can be inverted:

$$\tilde{D}^{\alpha\beta}(\mathbf{x}, m; z) = -\frac{D_0 \xi m^{z-\xi}}{z-\xi}\int \frac{d^d q}{(2\pi)^d}\frac{e^{\imath \mathbf{q}\cdot\mathbf{x}}}{q^{d+z}}\Pi^{\alpha\beta}(\hat{\mathbf{q}}). \tag{B.2}$$

The integral is absolutely convergent in the infra-red and is convergent in the ultra-violet because of the oscillatory exponential. Furthermore incompressibility constrains the tensorial structure of the integral to the form

$$\tilde{D}^{\alpha\beta}(\mathbf{x}, m; z) = \frac{\tilde{D}^{\gamma}{}_{\gamma}(\mathbf{x}; m)}{\mathcal{T}^{\mu}{}_{\mu}(\hat{\mathbf{x}}, z)} \mathcal{T}^{\alpha\beta}(\hat{\mathbf{x}}, z) \tag{B.3}$$

with $T^{\alpha\beta}$ defined by (15). Thus, the knowledge of the trace allows to reconstruct (19). The trace can be computed using the identity

$$\int \frac{d^d q}{(2\pi)^d} \frac{e^{\imath q\cdot x}}{q^{d+z}} = \frac{x^z}{2^z (4\pi)^{\frac{d}{2}}} \frac{\Gamma\left(-\frac{z}{2}\right)}{\Gamma\left(\frac{d+z}{2}\right)} \tag{B.4}$$

which is derived using (B.1). The result is

$$\frac{\tilde{D}^{\gamma}{}_{\gamma}(\mathbf{x}, m; z)}{\mathcal{T}^{\mu}{}_{\mu}(\hat{\mathbf{x}}, z)} = \frac{D\xi m^{z-\xi} x^z}{z(z-\xi)} \frac{(z+d-1)d}{(d-1)(d+z)} \frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(1-\frac{z}{2}\right)}{2^z \Gamma\left(\frac{d+z}{2}\right)} \tag{B.5}$$

with D now given by (11). Inserting (B.5) in (B.3) recovers (19).

An alternative derivation of (B.5) which does not use oscillatory integrals is available from Ref. 53.

C. FIRST ORDER APPROXIMATION

Rewriting (78) as the difference

$$\mathcal{V}^{\alpha\beta}_{(1;4)}(\mathbf{x}, m) = 2\frac{D_0 m^{\xi}}{D} \int_{q\geq m} \frac{d^d q}{(2\pi)^d} \frac{\Pi^{\alpha\beta}(\hat{\mathbf{q}})}{q^{d+2+\xi}} - 2\frac{D_0 m^{\xi}}{D} \int_{q\geq m} \frac{d^d q}{(2\pi)^d} \frac{e^{\imath q\cdot x}}{q^2} \frac{\Pi^{\alpha\beta}(\hat{\mathbf{q}})}{q^{d+\xi}} \tag{C.1}$$

permits to evaluate it using the same method expounded in appendix B. The Mellin transform of the second interval on the right hand side of (C.1) is well defined for $\Re z < -2$. However, the residue of the pole in $z = -2$ cancels exactly with the first integral on the right hand side of (C.1). Observing that (C.1) has vanishing divergence its Mellin transform can be finally written as

$$\tilde{\mathcal{V}}^{\alpha\beta}(\mathbf{x}, m; z+2) = \frac{\tilde{\mathcal{V}}^{\gamma}_{(1;4)\gamma}(\mathbf{x}; m, z+2)}{\mathcal{T}^{\mu}{}_{\mu}(\hat{\mathbf{x}}, z+2)} \mathcal{T}^{\alpha\beta}(\hat{\mathbf{x}}, z+2), \qquad -2 < \Re z < 0 \tag{C.2}$$

with

$$\frac{\tilde{\mathcal{V}}^{\gamma}_{(1;4)\gamma}(\mathbf{x}, m; z+2)}{\mathcal{T}^{\mu}{}_{\mu}(\hat{\mathbf{x}}; z+2)} = \frac{2(d+z+1)c(z)}{(z+2)(d+z-1)(d+2+z)} \frac{m^z x^{2+z}}{z(z-\xi)} \tag{C.3}$$

and $c(z)$defined by (20). Setting ξ to zero, (C.2) has a double pole at z equal zero. The amplitude of such pole specifies is equal to minus the prefactor of the logarithm in the short distance asymptotics of the diagram

$$\mathcal{V}^{(0)\alpha\beta}_{(1;4)}(\mathbf{x};m) = -\frac{(d+1)x^2\mathcal{T}^{\alpha\beta}(\hat{\mathbf{x}},2)}{(d-1)(d+2)}\left[\ln\left(\frac{mx}{2}\right) - \frac{\psi\left(\frac{d+4}{2}\right)+\psi(1)}{2}\right] + \frac{x^2\delta^{\alpha\beta}}{2(d+2)} + O(m^2x^2) \tag{C.4}$$

with

$$\psi(x) = \left.\frac{d}{du}\right|_{u=0} \ln\Gamma(u+x). \tag{C.5}$$

Differentiating the Mellin transform with respect to ξ higher orders contribution are found. In particular

$$\mathcal{V}^{(1)\alpha\beta}_{(1;4)}(\mathbf{x};m) = -\frac{(d+1)x^2\mathcal{T}^{\alpha\beta}(\hat{\mathbf{x}},2)}{2(d-1)(d+2)}\left[\ln\left(\frac{mx}{2}\right)\right]^2 + \left\{\frac{(d+1)x^2\mathcal{T}^{\alpha\beta}(\hat{\mathbf{x}},2)}{(d-1)(d+2)}\frac{\psi\left(\frac{d+4}{2}\right)+\psi(1)}{2} + \frac{x^2\delta^{\alpha\beta}}{2(d+2)}\right\}\ln\left(\frac{mx}{2}\right) + O(m^2x^2). \tag{C.6}$$

For more details the reader is referred to Ref. 53.

D. SECOND ORDER APPROXIMATION

The scope of this appendix is to expound the computational strategy followed in the evaluation of the second order diagrams and to give the final results used in the determination of the scaling exponents. Further details together with the computer packages used in the practical evaluation are available from ref. 53.

The evaluation of higher order integrals is hampered by the fact that the Mellin transform with respect of the spatial argument of the structure functions does not remove uniformly the mass cutoff from all the momentum integrations. The problem is obviated by taking the convolution of as many Mellin transforms as the number of momentum integration affected by the cutoff. The procedure can be illustrated by considering the general form of second order diagrams:

$$\tilde{\mathcal{V}}(\mathbf{x}, z+n) = \int_0^\infty \frac{dw}{w}\frac{1}{w^{z+n}}\int_{p,q>m}\frac{d^dp\,d^dq}{(2\pi)^{2d}}\frac{f(\mathbf{p},\mathbf{q},w\mathbf{x})}{q^d p^d}. \tag{D.1}$$

The integer n is fixed by the degree of homogeneity of the function f:

$$f(w\mathbf{p}, w\mathbf{q}, w\mathbf{x}) = w^n f(\mathbf{p},\mathbf{q},\mathbf{x}). \tag{D.2}$$

Thus the translation of the origin in the complex z-plane makes sure that the integration contour of the Mellin anti-transform poles lies to the left of the pole in z equal zero. By rescaling the integral becomes

$$\tilde{\mathcal{V}}(\mathbf{x}, z+n) = \int_0^\infty \frac{dw}{w} \frac{m^z}{w^z} \int_{p,q>w} \frac{d^d p d^d q}{(2\pi)^{2d}} \frac{f(\mathbf{p}, \mathbf{q}, \mathbf{x})}{q^d p^d}, \qquad z < 0. \tag{D.3}$$

Taking the convolution with a second Mellin transform allows to deal with unbounded momentum integrations:

$$\begin{aligned}\tilde{\mathcal{V}}(\mathbf{x}, z+n) &= \int_0^\infty \frac{dw}{w} \frac{m^z}{w^z} \int_{\substack{z<\Re\zeta<0 \\ -\infty<\Im\zeta<\infty}} \frac{d\zeta}{(2\pi\imath)} \int_0^\infty \frac{du}{u} \frac{1}{u^\zeta} \int_{p,q>w} \frac{d^d p d^d q}{(2\pi)^{2d}} \frac{f(\mathbf{p}, \mathbf{q}, \mathbf{x})}{q^d p^d} \\ &= \int_{\substack{z<\Re\zeta<0 \\ -\infty<\Im\zeta<\infty}} \frac{d\zeta}{(2\pi\imath)} \frac{m^z}{\zeta(z-\zeta)} \int \frac{d^d p d^d q}{(2\pi)^{2d}} \frac{f(\mathbf{p}, \mathbf{q}, \mathbf{x})}{q^{d+\zeta} p^{d+z-\zeta}}.\end{aligned} \tag{D.4}$$

Performing, eventually with the help of the representation of a power-law (B.1), the integration over momenta leaves with an integral over the Mellin variable ξ. In order to determine the scaling exponents up to second order, it is necessary to compute the Mellin transform of each diagram up to $\mathcal{O}(z^{-1})$accuracy. The observation allows for some simplifications. Namely dimensional considerations impose

$$\tilde{f}(\mathbf{x}, \zeta, z-\zeta) := \int \frac{d^d p d^d q}{(2\pi)^{2d}} \frac{f(\mathbf{p}, \mathbf{q}, \mathbf{x})}{q^{d+\zeta} p^{d+z-\zeta}} = x^{z+n} \tilde{f}(\hat{\mathbf{x}}, \zeta, z-\zeta) \tag{D.5}$$

with $\tilde{f}$ not vanishing for z equal zero. Thus, it is possible to infer the form of the Laurent expansion for z and ζ in the neighborhood of zero

$$\tilde{f}(\mathbf{x}, \zeta, z-\zeta) = \frac{x^{z+n}}{z} \left[\frac{\tilde{f}_{(-3;1)}(\hat{\mathbf{x}})}{\zeta} + \frac{\tilde{f}_{(-3;1)}(\hat{\mathbf{x}})}{z-\zeta} + \tilde{f}_{(-2)} + O(\zeta, z-\zeta) \right]. \tag{D.6}$$

The knowledge of the first two poles of $\tilde{f}$ around ζ equal zero suffices to reconstruct the first two terms of the Laurent series around z equal zero of $\tilde{\mathcal{V}}$:

$$\begin{aligned}\tilde{\mathcal{V}}(\mathbf{x}, z+n) = x_n (mx)^z \int_{\substack{z<\Re\zeta<0 \\ -\infty<\Im\zeta<\infty}} \frac{d\zeta}{2\pi} \frac{m^{-2}}{z\zeta(z-\zeta)} \\ \times \left[\frac{\tilde{f}_{(-3;1)}(\hat{\mathbf{x}})}{\zeta} + \frac{\tilde{f}_{(-3;2)}(\hat{\mathbf{x}})}{z-\zeta} + \tilde{f}_{(-2)} + O(\zeta, z-\zeta) \right]\end{aligned}$$

$$= x^n (mx)^z \left[\frac{\tilde{f}_{(-3;1)}(\hat{\mathbf{x}}) + \tilde{f}_{(-3;2)}(\hat{\mathbf{x}})}{z^3} + \frac{\tilde{f}_{(-2)}(\hat{\mathbf{x}})}{z^2} \right] + O(z^{-1}). \quad \text{(D.7)}$$

In essence, inner Mellin transforms work as regularized Taylor expansions for the argument of multi-loop integrals. The method was applied to the evaluation of $\mathcal{V}_{(2;4)}$ and $\mathcal{V}_{(2;6)}$. The integral $\mathcal{V}_{(2;8)}$ in the limit of infinite integral scale of the forcing reduces to the product of two first order integrals. The evaluation of its Mellin transform through convolutions can be used to check of the procedure.

D.1. Evaluation of $\mathcal{V}_{(2;4)}$

The diagram $\mathcal{V}^{\alpha\beta}_{(2;4)}$ is most conveniently evaluated if its real space representation

$$\mathcal{V}^{\alpha\beta}_{(2;4)} = \frac{2}{\xi} \int_0^\infty ds \int d^d y \frac{e^{-\frac{y^2}{4\varkappa_\star s}} - e^{-\frac{(\mathbf{x}-\mathbf{y})^2}{4\varkappa_\star s}}}{(4\pi\varkappa_\star s)^{\frac{d}{2}}} D^{\mu\nu}(\mathbf{y}; m)(\partial_\mu \partial_\nu \mathcal{V}^{\alpha\beta}_{(1;4)})(\mathbf{y}; m) \quad \text{(D.8)}$$

is adopted as a starting point for its evaluation. The choice of the prefactor guarantees that at zero molecular viscosity the interaction term is order is of the order $O(\xi^0)$. The reason is that in real space space (D.8) has the same structure of the first order vertex $\mathcal{V}^{\alpha\beta}_{(1;4)}$. Once the Mellin transform of this latter is known, the use of the Mellin convolution technique (appendix A) reduces the evaluation (D.8) to that of an integral similar of the same type of (78). More explicitly the Mellin transform is

$$\tilde{\mathcal{V}}^{\alpha\beta}_{(2;4)}(z+2) = \frac{2}{\xi} \int\limits_{\substack{z<\Re\zeta<0 \\ -\infty<\Im\zeta<\infty}} \frac{d\zeta}{2\pi} \int\limits_0^\infty ds \int d^d y \frac{e^{-\frac{y^2}{4\varkappa_\star s}} - e^{-\frac{(\mathbf{x}-\mathbf{y})^2}{4\varkappa_\star s}}}{(4\pi\varkappa_\star s)^{\frac{d}{2}}} \tilde{D}^{\mu\nu}(\mathbf{y}, m, \zeta)(\partial_\mu \partial_\nu \tilde{\mathcal{V}}^{\alpha\beta}_{(1;4)})(\mathbf{y}, m, z-\zeta) \quad \text{(D.9)}$$

The integrals over space-time variables can be performed exactly:

$$\tilde{\mathcal{V}}^{\alpha\beta}_{(2;4)}(z+2) = \int\limits_{\substack{z<\Re\zeta<0 \\ -\infty<\Im\zeta<\infty}} \frac{d\zeta}{2\pi} \times \frac{2^{-z} m^{z-2\xi} x^z \Gamma\left(\frac{2+d}{2}\right)^2 \Gamma\left(1-\frac{\zeta}{2}\right) \Gamma\left(\frac{2+\zeta-z}{2}\right) [P_0(z,\zeta,d) x^2 \delta^{\alpha\beta} - P_1(z,\zeta,d) x^\alpha x^\beta]}{(d-1)^2 \zeta(\zeta-z) z(2+z)(2+d+z)(\zeta-\xi)(\zeta-z+\xi) \Gamma\left(\frac{4+d+\zeta}{2}\right) \Gamma\left(\frac{2+d-\zeta+z}{2}\right)} \quad \text{(D.10)}$$

with

$$P_0(z,\zeta,d) = z[d^2(2+z)+d^3-3z-4-3d] + \zeta[2+d^2z-z^2+d(2+z+z^2)] \quad \text{(D.11a)}$$

$$P_1(z,\zeta,d) = (2+z)[d^2\zeta-(2+\zeta)z+d\zeta(1+z)]. \quad \text{(D.11b)}$$

In order to determine scaling exponents with second order accuracy it is enough to evaluate (D.10) at ξ equal zero. To that goal the integral over ξ can be performed by applying Cauchy theorem in the complex ξ plane. Since the contour is has clockwise orientation residues must multiplied by a minus sign. Logarithmic contributions to (D.8) are associated to the triple and double pole in z of the Mellin transform. These latter ones are fully specified if (D.10) is approximated by its first residue for ξ equal zero. The result can be couched into the form

$$\tilde{\mathcal{V}}^{(0)\alpha\beta}_{(2;4)}(\mathbf{x};z+2) = -\left(\frac{mx}{2}\right)^z\left\{\left[\frac{1}{z^3}-\frac{\psi\left(\frac{d+2}{2}\right)+\psi(1)}{2z^2}\right]V^{\alpha\beta}_{(2;4;1)}(\mathbf{x})+\frac{V^{\alpha\beta}_{(2;4;2)}(\mathbf{x})}{z^2}\right\}+O(z^{-1}) \quad \text{(D.12)}$$

with the function φ defined by (C.5). The tensor coefficients appearing in (D.12) are

$$\tilde{\mathcal{V}}^{\alpha\beta}_{(2;4;1)}(\mathbf{x}) := 2\frac{(d+1)(d^2+d-3)x^2\delta^{\alpha\beta}-(d^2+d-4)\mathbf{x}^\alpha\mathbf{x}^\beta}{(d-1)^2(d+2)^2} \quad \text{(D.13)}$$

and

$$\tilde{\mathcal{V}}^{\alpha\beta}_{(2;4;2)}(\mathbf{x}) := \frac{[4-d(d^3+4d^2+d-10)]x^2\delta^{\alpha\beta}-8\mathbf{x}^\alpha\mathbf{x}^\beta}{(d-1)^2(d+2)^3}. \quad \text{(D.14)}$$

The residue in z equal zero of (D.8) specifies the inertial range asymptotics of the diagram at leading order in ξ.

D.2. Evaluation of $\mathcal{V}_{(2;6)}$

The Mellin transform of $\mathcal{V}_{(2;6)}$ can be written as the sum of three terms

$$\tilde{\mathcal{V}}^{\alpha\beta;\mu}_{(2;6)}(z+3) = \sum_{i=1}^{3}\tilde{\mathcal{V}}^{\alpha\beta;\mu}_{(2;6;i)}(z+3) \quad \text{(D.15)}$$

with

$$\tilde{\mathcal{V}}^{\alpha\beta;\mu}_{(2;6;1)}(z+3) = \int_0^\infty \frac{dw}{w}\frac{D_0^2m^{z-2\xi}}{D^2w^{z-2\xi}}\int_{\substack{q\geq w\\ p\geq w}}\frac{d^dq\,d^dp}{(2\pi)^{2d}}\frac{2\sin(\mathbf{p}\cdot\mathbf{x})}{(q^2+\mathbf{q}\cdot\mathbf{p}+p^2)q^2}\frac{\mathbf{q}_\nu\Pi^{\mu\nu}(\hat{\mathbf{p}})\Pi^{\alpha\beta}(\hat{\mathbf{q}})}{p^{d+\xi}q^{d+\xi}}$$

$$\tilde{\mathcal{V}}^{\alpha\beta;\mu}_{(2;6;2)}(z+3) = \int_0^\infty \frac{dw}{w} \frac{D_0^2 m^{z-2\xi}}{D^2 w^{z-2\xi}} \int_{\substack{q\geq w\\ p\geq w}} \frac{d^d q d^d p}{(2\pi)^{2d}} \frac{2\sin(\mathbf{p}\cdot\mathbf{x})}{(q^2+\mathbf{q}\cdot\mathbf{p}+p^2)q^2} \frac{\mathbf{q}_\nu \Pi^{\mu\nu}(\hat{\mathbf{p}})\Pi^{\alpha\beta}(\hat{\mathbf{q}})}{p^{d+\xi}q^{d+\xi}}$$

$$\tilde{\mathcal{V}}^{\alpha\beta;\mu}_{(2;6;3)}(z+3) = -\int_0^\infty \frac{dw}{w} \frac{D_0^2 m^{z-2\xi}}{D^2 w^{z-2\xi}} \int_{\substack{q\geq w\\ p\geq w}} \frac{d^d q d^d p}{(2\pi)^{2d}} \frac{2\sin[(\mathbf{q}+\mathbf{p})\cdot\mathbf{x}]}{(q^2+\mathbf{q}\cdot\mathbf{p}+p^2)q^2} \frac{\mathbf{q}_\nu \Pi^{\mu\nu}(\hat{\mathbf{p}})\Pi^{\alpha\beta}(\hat{\mathbf{q}})}{p^{d+\xi}q^{d+\xi}} \tag{D.16}$$

In order to determine the scaling exponent within second order, ξ can be set to zero in the integrands. The three Mellin integrals exist separately in the complex z-plane for values of z such that $\Re z < -2$. The sum of the three integrals brings about the cancellations restoring the original domain of convergence of the Mellin transform of $\mathcal{V}_{(2;6)}$.

The explicit evaluation of the integrals is cumbersome but can be performed using some software for symbolic manipulations. The packages used for the evaluation are available for free download from Ref. 53. The final result is

$$\tilde{\mathcal{V}}^{(0)\alpha\beta;\mu}_{(2;6)}(\mathbf{x};z+3) = -\left(\frac{mx}{2}\right)^z \left\{ \left[\frac{1}{z^3} - \frac{\psi\left(\frac{2+d}{2}\right)+\psi(1)}{2z^2} \right] V^{\alpha\beta;\mu}_{(2;6;1)}(\mathbf{x}) + \frac{V^{\alpha\beta;\mu}_{(2;6;2)}(\mathbf{x})}{z^2} + O(z^{-1}) \right\}. \tag{D.17}$$

The tensor coefficients are

$$V^{\alpha\beta;\mu}_{(2;6;1)}(\mathbf{x}) := \frac{4x^\alpha x^\beta x^\mu + (d+1)x^2[(d-1)\delta^{\alpha\beta}x^\mu - \delta^{\alpha\mu}x^\beta - \delta^{\mu\beta}x^\alpha]}{(d-1)^2(d+2)^2} \tag{D.18}$$

$$V^{\alpha\beta;\mu}_{(2;6;2)}(\mathbf{x}) := v_{(1)}x^2 x^\mu \delta^{\alpha\beta} + v_{(2)}x^2(x^\beta\delta^{\alpha\mu} + x^\alpha\delta^{\mu\beta}) + v_{(3)}x^\alpha x^\beta x^\mu \tag{D.19}$$

where the scalar coefficients $\{v_{(i)}\}_{i=1}^3$ are

$$v_{(1)} = \frac{2(28+20d-15d^2-8d^3-d^4) - 3(12+5d-4d^2-d^3)\mathrm{Hyp}_{21}\left(1,1,2+\frac{d}{2},\frac{1}{4}\right)}{4(d-1)^2(d+2)^3(d+4)}$$

$$v_{(2)} = \frac{2(-4+4d+3d^2+d^3) + 3(4+3d+d^2)\mathrm{Hyp}_{21}\left(1,1,2+\frac{d}{2},\frac{1}{4}\right)}{4(d-1)^2(d+2)^3(d+4)} \tag{D.20}$$

$$v_{(3)} = -\frac{2(6+d+d^2) + 3d\,\mathrm{Hyp}_{21}\left(1,1,2+\frac{d}{2},\frac{1}{4}\right)}{(d-1)^2(d+2)^3(d+4)}.$$

D.3. Evaluation $\mathcal{V}_{(2,8)}$

As shown in the main text, when the integral scale of the forcing tends to infinity, this integral factorizes to

$$\mathcal{V}^{\alpha\beta;\mu\nu}_{(2,8)} = \mathcal{V}^{\alpha\beta}_{(1,4)}\mathcal{V}^{\mu\nu}_{(1,4)}. \tag{D.21}$$

Hence, the knowledge of the small scale asymptotics of the first order vertex $\mathcal{V}_{(1,4)}$ suffices to determine the one of $\mathcal{V}_{(2,8)}$. Scaling exponents are conveniently evaluated using the Mellin transform of diagrams. Noting that $\mathcal{V}_{(1,4)}$ has the form

$$\mathcal{V} = A(x)\ln x + B(x) \tag{D.22}$$

with A, B some cut-off functions having finite value in zero and vanishing at infinity, the Mellin transform of (D.21) can be represented as

$$\tilde{\mathcal{V}}^2(x,z) = x^z \int_{-\infty}^{\infty} due^{-u}\left[A^2(e^{\frac{u}{z}})\frac{u^2}{z^3} + 2B(e^{\frac{u}{z}})A(e^{\frac{u}{z}})\frac{u}{z^2} + \frac{B^2(e^{\frac{u}{z}})}{z}\right]. \tag{D.23}$$

For z tending to zero from below one finds

$$\lim_{z\uparrow 0}\tilde{\mathcal{V}}^2(x,z) = \int_0^{\infty} due^{-u}\left[A^2(0)\frac{u^2}{z^3} + 2B(0)A(0)\frac{u}{z^2} + \frac{B^2(0)}{z}\right] + O(z^0). \tag{D.24}$$

The equality entails that

$$\tilde{\mathcal{V}}^{(0)\alpha\beta\mu\nu}_{(2,8)}(\mathbf{x}z+4) = -\left(\frac{mx}{2}\right)^z\left\{\left[\frac{1}{z^3} - \frac{\psi\left(\frac{2+d}{2}\right)+\psi(1)}{2z^2}\right]V^{\alpha\beta;\mu\nu}_{(2;8;1)}(\mathbf{x}) + \frac{V^{\alpha\beta;\mu\nu}_{(2;8;2)}(\mathbf{x})}{z^2} + O(z^{-1})\right\} \tag{D.25}$$

with

$$V^{\alpha\beta;\mu\nu}_{(2;8;2)}(\mathbf{x}) := \frac{2(d+1)^2x^4\mathcal{T}^{\alpha\beta}(\hat{\mathbf{x}},2)\mathcal{T}^{\mu\nu}(\hat{\mathbf{x}},2)}{(d-1)^2(d+2)^2} \tag{D.26a}$$

$$V^{\alpha\beta;\mu\nu}_{(2;8;1)}(\mathbf{x}) := -\frac{d(3+4d+d^2)x^4\delta^{\alpha\beta}\delta^{\mu\nu} + 8x^\alpha x^\beta x^\mu x^\nu - (2+5d+d^2)x^2(\delta^{\mu\nu}x^\alpha x^\beta + \delta^{\alpha\beta}x^{mu}x^{nu})}{(d-1)^2(d+2)^3}. \tag{D.26b}$$

E. ANGULAR INTEGRALS

Let ω denote the azimuthal angle in a given reference frame. The projection of powers of $\cos\omega$ onto hyperspherical harmonics with zero magnetic numbers can be evaluated by considering the generating function

$$\sum_{n=0}^{\infty}\frac{(\imath z)^n}{n!}\int d\Omega Y^{*}_{j;0}(\Omega)\cos^n(\omega)=\int d\Omega Y_{j;0}(\Omega)e^{\imath z\cos(\omega)} \tag{E.1}$$

The exponential can be expanded in hyperspherical harmonics

$$e^{\,z\cos(\omega)}=\sum_{j=0}^{\infty}\frac{\imath^j N_{j,d}}{z^{\frac{d-2}{2}}}\mathrm{BesJ}\left(j+\frac{d-2}{2};z\right)Y_{j;0}(\Omega) \tag{E.2}$$

with $N_{j,d}$ normalization factor irrelevant for the present considerations. Since

$$\mathrm{BesJ}\left(j+\frac{d-2}{2};z\right)=\left(\frac{z}{2}\right)^{j+\frac{d-2}{2}}\sum_{k=0}^{\infty}\frac{\left(-\frac{z^2}{4}\right)^k}{\Gamma(k+1)\Gamma(j+\frac{d-2}{2}+k+1)} \tag{E.3}$$

angular integrals are just the n-th coefficient of the Taylor expansion

$$\int d\Omega Y^{*}_{j;0}(\Omega)\cos^n(\omega)=\frac{2^{1-\frac{d}{2}-n}N_{j,d}\Gamma(1+n)}{\Gamma\left(\frac{2-j+n}{2}\right)\Gamma\left(\frac{d+j+n}{2}\right)} \tag{E.4}$$

whence (197) follows.

F. GRADIENT EXPANSION INTEGRALS

The first order integral can be readily performed

$$\begin{aligned}\mu^{\alpha\beta}_{(1;4)}(z)&=\frac{D_0 m^{\xi}M^z}{zD}\int_{\infty>p\geq m}\frac{d^d p}{(2\pi)^d}\frac{(\mathbf{p}\cdot\mathbf{x})^2}{p^2}\frac{\Pi^{\alpha\beta}(\hat{\mathbf{p}})}{p^{d+z+\xi}}\\&=\frac{(d+1)x^2}{z(z+\xi)(d-1)(d+2)}\left(\frac{M}{m}\right)^2\mathcal{T}^{\alpha\beta}(\hat{\mathbf{x}},2).\end{aligned} \tag{F.1}$$

Second order integrals can be performed by resorting to the Mellin-convolution techniques expounded in the previous appendix D. Second order integrals can be always reduced to the scalar form

$$\mathcal{U}(M,m;z)=\int_{\substack{\Re\zeta<z\\-\infty<\Im\zeta<\infty}}\frac{d\zeta}{2\pi}\frac{M^z G(m,\zeta,z-\zeta)}{z\zeta(z-\zeta)} \tag{F.2}$$

with

$$G(m,\zeta,z-\zeta)=\int_m^\infty\int_m^\infty \frac{dpdq}{pq}\frac{\varphi(p,q)}{p^{z-\zeta}q^{zeta}} = m^z z\int_1^\infty \frac{dq}{q}\left[\frac{\varphi(1,q)}{q^\zeta}+\frac{\varphi(q,1)}{q^{z-\zeta}}\right] \tag{F.3}$$

and angular degrees of freedom re-absorbed into the definition of φ.

The integrals to perform are essentially those calculated in Ref. 3 when direct renormalization and dimensional regularization were applied to compute the scaling exponents of the Kraichnan model. The reader interested to the details of the calculations is therefore referred to Ref. 3. Here the results are presented in the

$$\begin{aligned}\mathcal{U}^{(0)\alpha\beta}_{(2;4)} &= \left(\frac{M}{m}\right)^z\left[\frac{V^{\alpha\beta}_{(2;4;1)}(\mathbf{x})}{z^3}+\frac{U^{\alpha\beta}_{(2;4;1)}(\mathbf{x})}{z^2}+O(z^{-1})\right]\\ \mathcal{U}^{(0)\alpha\beta;\mu}_{(2;6)} &= \left(\frac{M}{m}\right)^z\left[\frac{V^{\alpha\beta;\mu}_{(2;6;1)}(\mathbf{x})}{z^3}+\frac{U^{\alpha\beta;\mu}_{(2;6;1)}(\mathbf{x})}{z^2}+O(z^{-1})\right]\end{aligned} \tag{F.4}$$

where $V_{(2;4;1)}$ and $V_{(2;6;1)}$ were respectively defined in (D.13), (D.18) while

$$U^{\alpha\beta}_{(2;4;1)}(\mathbf{x}) := -2(d+1)\frac{x^2\delta^{\alpha\beta}-dx^\alpha x^\beta}{(d-1)^2(d+2)^3} \tag{F.5}$$

and

$$U^{\alpha\beta;\mu}_{(2;6;2)}(\mathbf{x}) := u_{(1)}x^\mu x^2\delta^{\alpha\beta}+\mu_{(2)}x^2(x^\beta\delta^{\alpha\mu}+x^\alpha\delta_{\mu\beta})+\mu_{(3)}x^\alpha x^\beta x^\mu \tag{F.6}$$

where the scalar coefficients $\{u_{(i)}\}_{i=1}^3$ are

$$\begin{aligned}u_{(1)} &= -3\frac{8(1+d)+(-12-5d+4d^2+d^3)\mathrm{Hyp}_{21}(1,1,2+\frac{d}{2},\frac{1}{4})}{4(d-1)^2(2+d)^3(4+d)}\\ u_{(2)} &= \frac{8(1+d)^2-3(4+3d+d^2)\mathrm{Hyp}_{21}(1,1,2+\frac{d}{2},\frac{1}{4})}{4(d-1)^2(2+d)^3(4+d)}\\ u_{(3)} &= \frac{-4-2d+2d^2+3d\mathrm{Hyp}_{21}(1,1,2+\frac{d}{2},\frac{1}{4})}{(d-1)^2(2+d)^3(4+d)}.\end{aligned} \tag{F.7}$$

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